Uncertainty Growth Estimation in UncertaintyAnalyzer

H. Castrup, Ph.D.
President, Integrated Sciences Group
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Background

The uncertainty, $u(t)$, in the bias ε_b of a subject parameter at time t elapsed since measurement ($t = 0$) is computed using the value of the initial measurement uncertainty, $u(0)$, and the reliability model for the parameter population. The basic concept is an extension of the ergodic theorem that states that the distribution of an infinite population of values at equilibrium is identical to the distribution of values attained by a single member sampled an infinite number of times.

The reliability (in-tolerance probability) of the subject parameter at time t is related to the parameter's uncertainty according to

$$R(t) = \int_{-L_1}^{L_2} f[\varepsilon_b(t)]\, d\varepsilon_b \,, \qquad (1)$$

where $f[\varepsilon_b(t)]$ is the probability density function for the parameter bias, $-L_1$ and L_2 are the parameter's tolerance limits. For discussion purposes, we will assume for the moment that the bias ε_b is normally distributed with the pdf given by

$$f[\varepsilon_b(t)] = \frac{1}{\sqrt{2\pi}\,u(t)} e^{-[\varepsilon_b - \mu(t)]^2/2u^2(t)} \,, \qquad (2)$$

where the variable $\mu(t)$ represents the parameter's expected bias at time t.

The relationship between L_1, L_2 and μ is shown below. Also shown is the distribution of the population of biases for the subject parameter of interest.

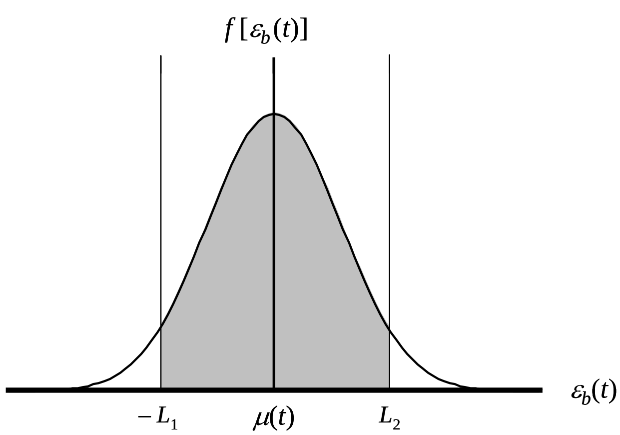

Figure 1. Probability density function for the subject parameter bias. The shaded area represents the in-tolerance probability at time *t*.

We state that at a given time t, the subject parameter's expected deviation from nominal is given by the relation

$$\mu(t) = \mu_0 + b(t) \,, \qquad (3)$$

where $b(0) = 0$.

At the time of measurement ($t = 0$), we estimate a value for μ_0 and label the uncertainty in this estimate $u(0)$. The remainder of this note discusses a method for calculating $u(t)$, given $u(0)$.

Uncertainty Growth Modeling

If we had at our disposal the reliability model for the individual measured parameter, given its initial uncertainty, we could obtain the uncertainty $u(t)$ in Eq. (1) directly, by iteration or by other means. However, we usually have information that relates only to the characteristics of the reliability model for the population to which the subject parameter belongs. This reliability model predicts the in-tolerance probability for the subject parameter population as a function of time elapsed since measurement. It can be thought of as a function that quantifies the *stability* of the population. In this view, we begin with a population in-tolerance probability at time $t = 0$ (immediately following measurement) and extrapolate to the in-tolerance probability at time $t > 0$.

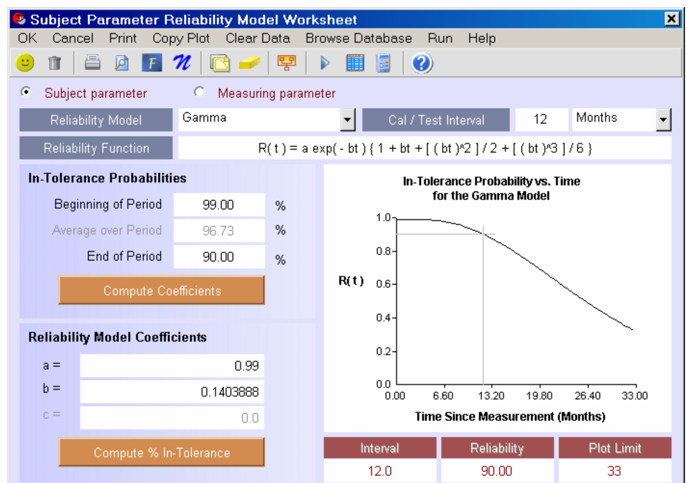

Figure 2. Reliability Modeling Example. The reliability model can be constructed using coefficients obtained from reliability analysis [1] or by entering a calibration interval, along with available in-tolerance probability data.

If we have recourse to a reliability modeling application, such as IntervalMAX [1], we can identify the appropriate reliability model and acquire the model's characteristics. This information can be entered directly in UncertaintyAnalyzer's Reliability Model Worksheet, as shown in Figure 2.. If we do not have recourse to the characteristics of the reliability model, we instead enter an elapsed time, a beginning-of-period (BOP) reliability and an end-of-period (EOP) reliability. For certain models, we must also enter an average-over-period (AOP) reliability. These values apply to the subject parameter's population and are based on service history records or engineering knowledge.

We next apply the reliability model obtained from these values to the individual parameter under consideration. In doing this, we operate under a set of assumptions.

Assumptions

In UncertaintyAnalyzer, using a population reliability model to estimate uncertainty growth for a parameter employs the following set of premises:

1. The result of a parameter measurement is an estimate of a parameter's value or bias. This result is accompanied by an estimate of the uncertainty in the parameter's bias.
2. The uncertainty of the measured parameter's bias or value at time $t = 0$ (immediately following measurement) is the uncertainty of the measurement process.[1]
3. The bias or value of the measured parameter is normally distributed around the measurement result.
4. The stability of the parameter is inferred from the stability of its population. This stability is represented by the population reliability model.
5. The uncertainty in the parameter's value or bias grows from its value at $t = 0$ in accordance with the reliability model of the parameter's population.[2]

Uncertainty Growth Estimation

As indicated above, uncertainty growth is estimated using the reliability model for the subject parameter. The expressions used to compute uncertainty growth vary depending on whether the parameter tolerances are two-sided, single-sided upper or single-sided lower.

General Two-Sided Cases

Using Eqs. (1) and (2) for parameters with two-sided tolerance limits, the reliability function at $t = 0$ is given by

$$\begin{aligned} R(0) &= \int_{-L_1}^{L_2} f[\varepsilon_b(0)]d\varepsilon_b \\ &= \frac{1}{\sqrt{2\pi}} \int_{-(L_1+\mu_0)/u_0}^{(L_2-\mu_0)/u_0} e^{-\zeta^2/2} d\zeta \\ &= \Phi\left(\frac{L_1+\mu_0}{u_0}\right) + \Phi\left(\frac{L_2-\mu_0}{u_0}\right) - 1, \end{aligned} \tag{4}$$

where the function $\Phi(\cdot)$ is defined by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\zeta^2/2} d\zeta ,$$

and where

$$u_0 \equiv u(0) .$$

The parameter μ_0 is an estimate of the parameter's bias at time $t = 0$, set equal to either a sample mean or a Bayesian estimate.[3] If μ_0 is set equal to a sample mean value, u_0 is set equal to the combined uncertainty estimate for the mean value. If μ_0 is set equal to a Bayesian estimate, u_0 is set equal to the uncertainty of the Bayesian estimate.

The reliability at time $t > 0$ is given by

$$R(t) = \Phi\left[\frac{L_1+\mu(t)}{u(t)}\right] + \Phi\left[\frac{L_2-\mu(t)}{u(t)}\right] - 1. \tag{5}$$

We use relations of the form of Eqs. (4) and (5) to estimate uncertainty growth. Since this growth consists of an increase in the initial uncertainty estimate, based on knowledge of the stability of the parameter population, it should not be influenced by the quantity $\mu(t)$. Accordingly, we construct two population reliability functions R_0 and R_t, defined by

$$R_0 = \Phi\left(\frac{L_1}{u_0}\right) + \Phi\left(\frac{L_2}{u_0}\right) - 1 , \tag{6}$$

and

$$R_t = \Phi\left(\frac{L_1}{u_t}\right) + \Phi\left(\frac{L_2}{u_t}\right) - 1 . \tag{7}$$

Next, we solve for u_0 and u_t iteratively. In UncertaintyAnalyzer this is done using the bisection method.[4]

Having obtained the solutions, we write

$$u(t) = u(0)\frac{u_t}{u_0} . \tag{8}$$

Once we obtain $u(t)$, we can then solve for the in-tolerance probability at time t by using Eq. (5).

At this point, we need a "best" estimate for μ. For this, we use Eq. (3). In applying this relation, we are cognizant of the fact that the uncertainty at time t is[5]

$$u(t) \to \sqrt{u^2(0) + u^2[b(t)]} . \tag{9}$$

If the function $b(t)$ is not known, we use the last known value of μ, namely μ_0, the value obtained by measurement. Substituting μ_0 for μ in Eq. (5) we have

$$R(t) \cong \Phi\left[\frac{L_1+\mu_0}{u(t)}\right] + \Phi\left[\frac{L_2-\mu_0}{u(t)}\right] - 1. \tag{10}$$

Restricted Two-Sided Cases

In cases where $\mu_0 = 0$ and $L_1 = L_2 = L$, Eqs. (6) and (7) becomes

[1] This may include an additional uncertainty due to error introduced by parameter adjustment or correction.

[2] See Eq. (1).

[3] In UncertaintyAnalyzer the Bayesian method is referred to as *SMPC* (Statistical Measurement Process Control). The method is discussed in the User Manual and described in References 5 – 11.

[4] See Chapter 9 of Press, et al., *Numerical Recipes in Fortran*, 2nd Ed., Cambridge University Press, 1992.

[5] Methods for determining $b(t)$ and $u[b(t)]$ are the subject of current research by Integrated Sciences Group.

$$R_0 = 2\Phi\left(\frac{L}{u_0}\right) - 1$$

and

$$R_t = 2\Phi\left(\frac{L}{u_t}\right) - 1 \ ,$$

which yield

$$u_0 = \frac{L}{\Phi^{-1}\left(\frac{1+R_0}{2}\right)}$$

and

$$u_t = \frac{L}{\Phi^{-1}\left(\frac{1+R_t}{2}\right)} ,$$

which, by Eq. (8), yields

$$u(t) = u(0)\frac{\Phi^{-1}\left(\frac{1+R_0}{2}\right)}{\Phi^{-1}\left(\frac{1+R_t}{2}\right)} .$$

Single-Sided Cases

In cases where tolerances are single-sided, $u(t)$ can be determined without iteration. In these cases, either L_1 or L_2 is infinite, and Eqs. (6) and (7) become

$$R_0 = \Phi\left(\frac{L \pm \mu}{u_0}\right)$$

and

$$R_t = \Phi\left(\frac{L \pm \mu}{u_t}\right),$$

where L is equal to L_1 for single-sided lower cases and is equal to L_2 for single-sided upper cases. The plus (minus) sign applies to single-sided lower (upper) cases.

Solving for $u(t)$ yields

$$u(t) = u(0)\frac{\Phi^{-1}(R_0)}{\Phi^{-1}(R_t)} .$$

Example: $\mu_0 = 0$, $R(t) = R(0)e^{-\lambda t}$:

$$R(0) = \Phi\left(\frac{L}{u_0}\right)$$

$$R(0)e^{\lambda t} = \Phi\left(\frac{L}{u_t}\right)$$

and

$$u(t) = u(0)\frac{\Phi^{-1}\left[R(0)\right]}{\Phi^{-1}\left[R(0)e^{-\lambda t}\right]} .$$

Confidence Limits

As in other UncertaintyAnalyzer functions, we compute confidence limits (expanded uncertainties) from a standard uncertainty, a confidence level and the degrees of freedom. With regard to uncertainty growth, $(1 - \alpha) \times 100\%$ confidence limits for the reported value μ_0 are constructed according to

$$\mu_0 \pm t_{\alpha/2,\nu} u(t) ,$$

where ν is the degrees of freedom for $u(0)$ and $t_{\alpha/2,\nu}$ is the corresponding t-statistic.

Supplement

Note that we have made no attempt to modify the degrees of freedom to take into account the fact that time has passed since the value $u(0)$ was obtained. Once methods are developed for estimating $b(t)$, this modification will follow naturally. For example, if we can model $b(t)$ according to

$$b(t) = \lambda t ,$$

then the parameter λ can be found by regression analysis, and the degrees of freedom become[6]

$$\nu_{total} = \frac{u^4(t)}{\frac{u^4(0)}{\nu} + \frac{s^4(\lambda)}{n-1}} ,$$

where $s(\lambda)$ is the standard uncertainty in the regression fit for λ and n is the number of observed values of $b(t) = \mu(t) - \mu_0$ employed in the analysis.

Caution

The use of $s(\lambda)$ in the expression for the total degrees of freedom is strictly justified only if values of $b(t)$ are computed for the observed data. If values of $b(t)$ are predicted for future calibrations, the uncertainty $u(t)$ in Eq. (9) must take into account an additional term that reflects a contribution due to the inherent growth in the uncertainty in $b(t)$.

This “inherent” contribution is the growth in uncertainty that exists in addition to the standard uncertainty in the regression. It arises from randomizing effects that are present during the time elapsed since calibration. Such effects are usually viewed as occurring in response to stresses due to usage, handling, storage and other factors.

Regression analysis will capture these effects for the particular sample used in the regression fit, but will not necessarily be applicable in making predictions outside this sample. Methods for modeling the inherent growth in $b(t)$ are the subject of current research.

References

[1] IntervalMAX, © 1994-2007, Integrated Software Associates, Distributed by Integrated Sciences Group, Bakersfield, CA 93306.

[2] ISO, *Guide to the Expression of Uncertainty in Measurement,* Geneva, Switzerland, 1995.

[4] Castrup, H., “Estimating Category B Degrees of Freedom,” *Proc. Measurement Science Conference*, Anaheim, January 2000.§

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[6] Jackson, D.H., "Analytical Methods to be Used in the Computer Software for the Manometer Audit System," SAIC/MED-TR-

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[6] The total degrees of freedom are obtained using the Welch-Satterthwaite relation, as recommended in Reference [2].

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[14] Castrup, H. and Castrup, S., "Calibration Scenarios for Uncertainty Analysis," Submitted for inclusion in *Proc. NCSLI Workshop & Symposium*, August 2008, Orlando.§