

# A Comprehensive Comparison of Uncertainty Analysis Tools

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## ABSTRACT

In recent years, compliance with ISO/IEC 17025, as well as other testing and calibration standards, has elevated the importance of estimating and reporting measurement uncertainty. ISO/TAG4/WG3 (the GUM) and ANSI/NCSL Z540-2-1997 [1] (the U.S. version of the GUM) provide guidelines for conducting an uncertainty analysis. Unfortunately, the implementation of these guidelines can be a daunting task, especially if one is not conversant in the necessary mathematical and statistical concepts. Consequently, testing and calibration personnel must often find off-the-shelf tools that meet their analysis requirements.

This paper presents a review and comparison of a number of software applications, both commercial and freeware, that have been developed in the past several years. Methodology, functionality, user-friendliness, documentation, technical support and other key criteria are addressed. Recommended guidelines for selecting an uncertainty analysis tool are also provided.

## INTRODUCTION

While the GUM provides general rules for analyzing and communicating measurement uncertainty, it does not focus on providing step-by-step instructions for evaluating specific measurement processes. As a result, testing and calibration personnel may find it difficult or confusing to apply these principles to specific measurement scenarios. In an effort to alleviate the challenging task of evaluating measurement uncertainty, various analysis tools have been developed in recent years and more are sure to follow.

Unfortunately, selecting an analysis tool can often be equally as frustrating and confusing as evaluating measurement uncertainty. As a developer and marketer of uncertainty analysis tools, it is important for Integrated Sciences Group to periodically assess the capabilities of similar software applications. The results of our software assessments are presented herein in an attempt to advance the collective knowledge and understanding in the measurement science and metrology community.

## PRACTICAL CONSIDERATIONS

Before we address the features and capabilities of individual software applications or tools, it is useful to first discuss some basic uncertainty analysis concepts [1-5]. Having a good understanding of these key concepts provides the basis for evaluating off-the-self analysis tools. The general uncertainty analysis procedure consists of the following steps:

1. Define the Measurement Process
2. Develop the Error Model
3. Identify the Error Sources and Distributions
4. Estimate Uncertainties
5. Combine Uncertainties
6. Report the Analysis Results

### Define the Measurement Process

The first step in any uncertainty analysis is to identify the physical quantity whose value is estimated via measurement. This quantity may be a directly measured value, such as the weight of a 1 gm mass or the output of a voltage reference. Alternatively, the quantity may be indirectly determined through the measurement of other variables, as in the case of estimating the volume of a cylinder by measuring its length and diameter. The former type of measurements are called

“direct measurements”, while the latter are called “multivariate measurements.”

For multivariate measurements, it is important to develop an equation that defines the mathematical relationship between the quantity of interest and the measured variables. For the cylinder volume example, this mathematical equation would be expressed as

$$V = \pi \left( \frac{D}{2} \right)^2 L \quad (1)$$

where

- $V$  = cylinder volume in units of interest
- $D$  = cylinder diameter in appropriate units
- $L$  = cylinder length in appropriate units
- $\pi$  = the ratio of the circumference of a circle to its diameter = 3.14159

At this stage of the analysis, it is also useful to briefly describe the test setup, environmental conditions, technical information about the instruments, reference standards, or other equipment used and the procedure for obtaining the measurement(s). This information will help identify the measurement process errors.

### Develop the Error Model

An error model is an algebraic expression that defines the total error in the value of a quantity in terms of all relevant measurement process or component errors. Errors in the length and diameter measurements contribute to the overall error in the estimation of the cylinder volume. Therefore, the cylinder volume equation can be expressed as

$$V_0 + \varepsilon_V = \pi \left( \frac{D_0 + \varepsilon_D}{2} \right)^2 (L_0 + \varepsilon_L) \quad (2)$$

where

- $V_0$  = true or nominal cylinder volume
- $D_0$  = true or nominal cylinder diameter
- $L_0$  = true or nominal cylinder length
- $\varepsilon_V$  = error in the cylinder volume measurement
- $\varepsilon_D$  = error in the cylinder diameter measurement
- $\varepsilon_L$  = error in the cylinder length measurement

By rearranging equation (2), we obtain an algebraic expression for the cylinder volume error.

$$\begin{aligned} \varepsilon_V &= \pi \left( \frac{D_0 + \varepsilon_D}{2} \right)^2 (L_0 + \varepsilon_L) - V_0 \\ &= \pi \left( \frac{D_0 + \varepsilon_D}{2} \right)^2 (L_0 + \varepsilon_L) - \pi \left( \frac{D_0}{2} \right)^2 L_0 \\ &= \frac{\pi}{4} (D_0^2 + 2D_0\varepsilon_D + \varepsilon_D^2) (L_0 + \varepsilon_L) - \pi \left( \frac{D_0}{2} \right)^2 L_0 \\ &= \frac{\pi}{4} (L_0 D_0^2 + 2L_0 D_0 \varepsilon_D + L_0 \varepsilon_D^2 + D_0^2 \varepsilon_L + 2D_0 \varepsilon_D \varepsilon_L + \varepsilon_D^2 \varepsilon_L) - \pi \left( \frac{D_0}{2} \right)^2 L_0 \end{aligned} \quad (3)$$

The terms,  $L_0 \varepsilon_D^2$ ,  $2D_0 \varepsilon_D \varepsilon_L$  and  $\varepsilon_D^2 \varepsilon_L$ , are referred to as second order terms and are considered to be small compared to the other first order terms in equation (3). Neglecting these terms, we can express the cylinder volume error model or equation in a simpler form.

$$\varepsilon_V = \frac{\pi}{4} (L_0 D_0^2 + 2L_0 D_0 \varepsilon_D + D_0^2 \varepsilon_L) - \pi \left( \frac{D_0}{2} \right)^2 L_0 \quad (4)$$

Rearranging equation (4), we can further simplify the equation for  $\varepsilon_V$ .

$$\begin{aligned}\varepsilon_V &= \pi \left( \frac{D_0}{2} \right)^2 \varepsilon_L + \pi L_0 \frac{D_0}{2} \varepsilon_D + \pi \left( \frac{D_0}{2} \right)^2 L_0 - \pi \left( \frac{D_0}{2} \right)^2 L_0 \\ &= \pi \left( \frac{D_0}{2} \right)^2 \varepsilon_L + \pi L_0 \frac{D_0}{2} \varepsilon_D\end{aligned}\quad (5)$$

The coefficients for  $\varepsilon_L$  and  $\varepsilon_D$  in equation (5) are actually the partial derivatives of  $V$  with respect to  $L$  and  $D$ .

$$\frac{\partial V}{\partial L} = \pi \left( \frac{D}{2} \right)^2 \quad \text{and} \quad \frac{\partial V}{\partial D} = \pi L \left( \frac{D}{2} \right)$$

Therefore, the cylinder volume error can be expressed as

$$\varepsilon_V = \left( \frac{\partial V}{\partial L} \right) \varepsilon_L + \left( \frac{\partial V}{\partial D} \right) \varepsilon_D \quad (6)$$

where the partial derivatives are **sensitivity coefficients** that determine the relative contribution of the errors in length and diameter to the total error.

### Identify Error Sources and Distributions

The errors in the cylinder length and diameter are the sum of the errors encountered during the measurement process and can be expressed as

$$\begin{aligned}\varepsilon_L &= \varepsilon_{L1} + \varepsilon_{L2} + \dots + \varepsilon_{Ln} \\ \varepsilon_D &= \varepsilon_{D1} + \varepsilon_{D2} + \dots + \varepsilon_{Dn}\end{aligned}$$

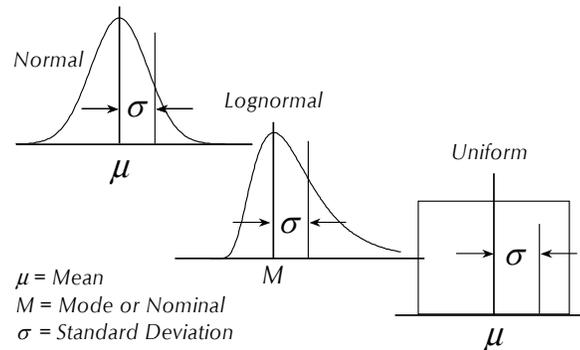
where the numbered subscripts signify the different measurement process errors. The errors most often encountered in making measurements include, but are not limited to the following:

- Measurement Bias - The bias in the measuring device and/or the quantity being measured.
- Random or Repeatability Error - The error associated with repeat measurements.
- Resolution Error - The error resulting from the finite resolution of the measuring device and/or the quantity being measured.
- Digital Sampling Error - The error introduced by digitizing an analog signal.
- Computation Error - The error due to round-off or computer truncation, numerical interpolation, empirically determined equations, etc.
- Operator Bias - The error introduced by the person making the measurements.
- Environmental Factors Error - The error introduced by variations in environmental conditions or by correcting for environmental conditions.
- Stress Response Error - Additional error resulting from stresses incurred during shipping and handling to the end-user after calibration.

**Measurement process errors are the basic elements of uncertainty analysis. Once these fundamental error sources have been identified, we can begin to develop uncertainty estimates.**

Another important aspect of the uncertainty analysis process is the fact that measurement errors can be characterized by probability distributions. The probability distribution for a type of measurement error is a mathematical description that relates the frequency of occurrence of values with the values themselves. Error distributions include, but are not limited to Normal, Lognormal, Uniform (rectangular), Triangular, Quadratic, Cosine, Exponential, U-Shaped, Trapezoidal and Student's t [4].

Each distribution is characterized by a set of statistics. The statistics most often used in uncertainty analysis are the mean, or the mode, and the standard deviation. With the lognormal distribution, a limiting value and the median value are also used.



**Figure 1 Error Distributions and Statistics**

In general, the three error distributions that have been found to be relevant to most real world measurement applications are the normal, lognormal and Student's t distributions. Other distributions such as the uniform, triangular, quadratic, trapezoidal and U-shaped have more specialized application. Details regarding error distributions and their application can be found in reference [4].

## Estimate Uncertainties

As previously stated, all measurements are accompanied by error. Our lack of knowledge about the sign and magnitude of measurement error is called measurement uncertainty. To better understand the relationship between measurement error and measurement uncertainty, we will discuss three important axioms that form the basis upon which uncertainties can be estimated. We will also review the variance addition rule, which provides a method for correctly combining uncertainties from different error sources.

**Axiom 1 - The uncertainty in a measured value is equal to the uncertainty in the measurement error.** This statement can be shown to be true from the following steps:

1. By definition, measurement error is the difference between the measured value and the true value. Conversely, the measured value is equal to the true value plus the measurement error.

$$\text{Measured Value} = \text{True Value} + \text{Measurement Error}$$

2. We define the function for uncertainty in the value  $x$  as

$$\text{Uncertainty}(x) = \text{Uncertainty in } x$$

3. The uncertainty in the measured value can then be expressed as

$$\text{Uncertainty (Measured Value)} = \text{Uncertainty (True Value)} + \text{Uncertainty (Measurement Error)}$$

But, the uncertainty in the true value is zero, so

$$\text{Uncertainty (Measured Value)} = \text{Uncertainty (Measurement Error)}$$

**Axiom 2 - Measurement errors follow probability distributions.** This statement basically indicates that errors can be described in such a way that their sign and magnitude have some definable probability of occurrence. With a basic understanding of error distributions and their statistics, we can estimate uncertainties. We begin with the statistical quantity called the variance, which is defined as the mean square dispersion of the distribution about its mean or mode value.

$$\text{var}(x) = \text{Mean Square Dispersion in } x$$

If a variable  $x$  follows a probability distribution, described by a probability density function  $f(x)$ , then the mean square dispersion or variance of the distribution [6] is given by

$$\text{var}(x) = \int_{-\infty}^{\infty} (x - \mu_x)^2 f(x) dx \tag{7}$$

where  $\mu_x$  is the mean of  $x$ . Because of the form of this definition, the variance is also referred to as the mean square error.

If a quantity  $x$  is a random variable representing a population of measurements, then the variance in  $x$  is just the variance in the error in  $x$ , which is expressed by the symbol  $\epsilon_x$ .

$$\text{var}(x) = \text{var}(x_{\text{true}} + \epsilon_x) = \text{var}(\epsilon_x) \tag{8}$$

**Axiom 3 - The uncertainty,  $u$ , in a measurement is the square root of the variance in the measurement error.**

If  $x$  is a measured value, then we can write

$$u_x = \sqrt{\text{var}(x)} = \sqrt{\text{var}(\epsilon_x)} \tag{9}$$

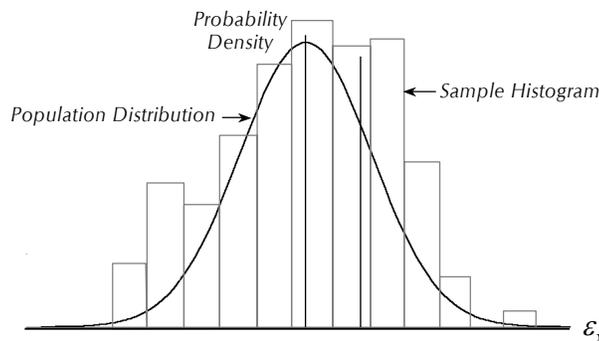
Axiom 3 provides the link between measurement error and measurement uncertainty. There are two approaches to estimating variance and uncertainty. Type A estimates involve data sampling and analysis. Type B estimates use engineering knowledge or recollected experience of measurement processes.

**Type A Estimates**

A Type A uncertainty estimate is defined as an estimate obtained from a sample of data. Data sampling involves making repeat measurements of the quantity of interest. It is important that each repeat measurement is independent, representative and taken randomly.

Random sampling is a cornerstone for obtaining relevant statistical information. Thus, Type A estimates usually apply to the uncertainty in repeatability or random error.

Because the data sample is drawn from a population of values, we make inferences about the population from certain sample statistics and from assumptions about the way the population of values is distributed. A sample histogram can aid in our attempt to picture the population distribution.



**Figure 2 - Random Error Distribution**

The normal distribution is ordinarily assumed to be the underlying distribution for random errors. When samples are taken, the sample mean and the sample standard deviation are computed and assumed to represent the mean and standard deviation of the population distribution. However, this equivalence is only approximate. To account for this,

the Student's t distribution is used in place of the normal distribution in computing confidence limits around the sample mean.

The **sample mean** can be thought of as an estimate of the value that we expect to get when we make a measurement. This "expectation value" is called the population mean, which is expressed by the symbol  $\mu$ . The sample mean,  $\bar{x}$ , is obtained by taking the average of the sampled values. The average value is computed by summing the values sampled and dividing them by the sample size,  $n$ .

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \dots + x_n) = \frac{1}{n} \sum_{i=1}^n x_i \quad (10)$$

The **sample standard deviation** provides an estimate of how much the population is spread about the mean value. The sample standard deviation,  $s_x$ , is computed by taking the square root of the sum of the squares of sampled deviations from the mean divided by the sample size minus one.

$$s_x = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2} \quad (11)$$

The value  $n-1$  is the **degrees of freedom** for the estimate, which signifies the number of independent pieces of information that go into computing the estimate. The greater the degrees of freedom, the closer the sample estimate will be to its population counterpart. The degrees of freedom for an uncertainty estimate is useful for establishing confidence limits and other decision variables.

We have already stated that the sample standard deviation is an estimate of the uncertainty in a value drawn randomly from its population. However, if the estimate is to represent the uncertainty in the mean value rather than the uncertainty in a single measurement, then the **uncertainty in the mean value** should be used. The uncertainty in the mean value,  $s_{\bar{x}}$ , is equal to the standard deviation divided by the square root of the sample size.

$$s_{\bar{x}} = \frac{s_x}{\sqrt{n}} \quad (12)$$

Once estimates of the sample mean and standard deviation have been obtained, and the degrees of freedom have been noted, it becomes possible to compute limits that bound the sample mean with some specified level of confidence. These limits are called **confidence limits** and the degree of confidence is called the **confidence level**.

Confidence limits can be expressed as multiples of the sample standard deviation. For normally distributed samples, this multiple is called the **t-statistic**. The value of the t-statistic is determined by the desired percent confidence level,  $C$ , and the degrees of freedom,  $\nu$ , for the sample standard deviation. Confidence limits around the sample mean are given by

$$\bar{x} \pm t_{\alpha/2, \nu} \frac{s_x}{\sqrt{n}} \quad (13)$$

where  $\alpha = (1 - C/100)$  and  $\nu = n - 1$ .

### Type B Estimates

In some cases, we must attempt to quantify the statistics of measurement error distributions by drawing on our recollected experience concerning the values of measured quantities or on our knowledge of the errors in these quantities. Estimates made in this manner are called heuristic or Type B estimates.

Conceivably, a Type B uncertainty estimate could be obtained by just "winging it." The problem is, that most of us do not have a point of reference for abstract quantities such as standard deviations or uncertainties. At best, we have a range of values that we have experienced or are able to surmise. The limiting values that bound these ranges are called **containment limits**. These limits can be viewed as bounding either measured values or measurement errors.

$x$  lies within  $\pm L$

In working with Type A estimates, we start with sample statistics and work toward developing confidence limits that bound values of a population with a specified confidence level or probability. In making Type B estimates, we apply this process in reverse. We begin with containment limits and a **containment probability**, estimate the degrees of freedom, and use these quantities to estimate the standard deviation or uncertainty<sup>1</sup>.

$x$  lies within  $\pm L$  with  $C\%$  confidence or probability

Containment limits may be estimated from experience or taken from some documented reference, such as manufacturer tolerance limits, stated expanded uncertainties obtained from calibration records or certificates, or statistical process control limits. Containment probability can be obtained from service history data, for example, as the number of observed in-tolerances,  $n_{\text{in-tol}}$ , divided by the number of calibrations,  $N$ .

$$C\% = 100\% \frac{n_{\text{in-tol}}}{N}$$

If a heuristic estimate is obtained solely from containment limits and containment probabilities, then the degrees of freedom is usually taken to be infinite. For example, if the measurement error is normally distributed, the uncertainty is computed from the containment limits,  $\pm L$ , the inverse normal distribution function,  $\Phi^{-1}(\cdot)$ , and the containment probability,  $p = C/100$ . The inverse normal distribution function can be found in statistics texts and in most spreadsheet programs. The appropriate relation is given below.

$$u = \frac{L}{\Phi^{-1}\left(\frac{1+p}{2}\right)} \quad (14)$$

If there is an uncertainty in the containment limits (e.g.,  $\pm L \pm \Delta L$ ) or the containment probability (e.g.,  $\pm p \pm \Delta p$ ), then it becomes imperative to estimate the **degrees of freedom**. As with Type A uncertainty estimates, the degrees of freedom quantifies the amount of information that goes into the Type B uncertainty estimate and is useful for establishing confidence limits and other decision variables.

Annex G of the GUM provides a relationship for computing the degrees of freedom for a Type B uncertainty estimate

$$\nu \approx \frac{1}{2} \frac{u^2(x)}{\sigma^2[u(x)]} \approx \frac{1}{2} \left[ \frac{\Delta u(x)}{u(x)} \right]^{-2} \quad (15)$$

where  $\sigma^2[u(x)]$  is the variance in the uncertainty estimate,  $u(x)$ , and  $\Delta u(x)$  is the relative uncertainty in the uncertainty estimate<sup>2</sup>. Hence, the degrees of freedom for a Type B estimate is inversely proportional to the square of the ratio of the uncertainty in the uncertainty divided by the uncertainty.

While this approach is intuitively appealing, the GUM offers no advice about how to determine  $\sigma^2[u(x)]$  or  $\Delta u(x)$ . Fortunately, since the publication of the GUM, a methodology for determining  $\sigma^2[u(x)]$  and computing the degrees of freedom for Type B estimates has been developed [5].

Once the containment limits, containment probability and the degrees of freedom have been established, we can estimate the standard deviation or uncertainty of the distribution of interest. For instance, if the measurement errors are normally distributed, we can construct a  $t$ -statistic based on the containment probability and degrees of freedom. The uncertainty estimate is then obtained by dividing the containment limit by the  $t$ -statistic, according to equation (16).

<sup>1</sup> Containment limits and containment probability are analogous to confidence limits and confidence level determined from Type A estimates.

<sup>2</sup> This equation assumes that the underlying error distribution is normal.

$$u = \frac{L}{t_{\alpha/2, \nu}} \quad (16)$$

### Estimating Measurement Process Uncertainties

Now that we are sufficiently versed in the statistics of error distributions and uncertainty estimates, we can discuss methods for determining the uncertainties associated with the measurement process. As previously discussed, measurement processes are characterized by errors from the following sources:

- What is being measured – measurand (subject parameter) bias and resolution error
- What it is being measured with – measurement reference (measuring parameter) bias and resolution error
- Who is making the measurement - operator bias
- How the measurement is made - digital sampling and computation errors
- Where the measurement is made - random and environmental factors errors
- What happens after the measurement - stress response error

**Bias uncertainty** is usually estimated heuristically via containment limits and containment probabilities. In some instances, as with reference standards, the stated parameter value and its associated uncertainty estimate may be supplied by a higher-level calibration laboratory. In this case, the bias uncertainty might have been obtained from statistical analysis of the calibration data and associated measurement process errors.

**Random uncertainty** is estimated from a statistical analysis of a sample of measurements. The estimated uncertainty is equal to the sample standard deviation. While the sample standard deviation is an estimate of the uncertainty in a value drawn randomly from its population, we often want to estimate and report the uncertainty in the mean value of a random sample. The uncertainty in the mean value is the sample standard deviation divided by the square root of the sample size.

**Resolution uncertainty** due to the precision of a reading or output is usually determined heuristically. The containment limits are based on the resolution of the device and an estimated containment probability. The distribution for resolution errors depends on whether the reading or output is expressed in analog or digital format. For **analog displays**, the resolution error is assumed to have a normal distribution. The resolution uncertainty is computed by setting the containment limits equal to a smallest increment of resolution and applying a containment probability that readings can be discerned within these limits. For **digital displays**, the resolution error is assumed to have a uniform distribution. The containment limits are  $\pm$  half the smallest displayed digit and the containment probability is equal to 100%.

**Digital sampling uncertainty** estimation involves defining a representative "signal" to be sampled and specifying a sampling rate, a sampling aperture time, a quantization precision (bits), an impulse response and hysteresis, a sampling noise level, and a sensor bias uncertainty. It also involves deciding on a model or methodology for eventual conversion of digitized data back to analog form. This multi-faceted analysis process is difficult without a structured template approach.

**Computation uncertainty** can result from round-off or computer truncation error, the linear interpolation of tabulated values, or using curve fit equations. Computation uncertainty is usually determined heuristically. The uniform distribution is applicable for round-off or computer truncation error. The triangular distribution is appropriate for describing errors resulting from linear interpolation or curve fit equations.

**Operator bias uncertainty** resulting from error in the perception of a human operator may be determined statistically, but is usually estimated heuristically. The usual way is to link it to resolution uncertainty or some other aspect of a measurement that can influence operator perception. The underlying error distribution is typically normal, but in some cases, the uniform distribution may apply.

**Environmental factors uncertainty** is typically estimated heuristically in a three-step process:

1. Estimate the uncertainty in each environmental/ancillary process error.
2. Multiply each environmental/ancillary uncertainty by its respective interaction coefficient.

- Combine the environmental/ancillary uncertainties, accounting for all correlations.

The **interaction coefficient** relates an environmental/ancillary factor to an error source. For example, if the error source under consideration is the measurement of length and the environmental factor is temperature, then the interaction coefficient is the thermal expansion coefficient for length.

**Stress response uncertainty** resulting from shipping and handling is usually estimated heuristically. The method of determination involves attempting to estimate the uncertainty in the stresses encountered and multiplying this uncertainty by a stress response coefficient (i.e., the response of the parameter value to stress). Stress response uncertainty should be considered, for example, when an item or device has been calibrated in an external laboratory to account for any expected increase in the uncertainty reported on the calibration certificate due to stresses incurred during transport back to the end-user. If a calibration laboratory already includes stress response uncertainty, it should be indicated in the uncertainty budget (i.e., the itemized list of error sources).

### Combine Uncertainties

For purposes of illustration, let us consider a quantity or parameter  $z$  that is obtained indirectly from the measurement of the quantities  $x$  and  $y$ . We will say that  $z$  is a linear function of the quantities  $x$  and  $y$

$$z = ax + by$$

where the coefficients  $a$  and  $b$  are constants.

In this case, we are interested in the uncertainty in  $z$  in terms of the uncertainties in the measured quantities  $x$  and  $y$ . Additionally, measurement errors for  $x$  and  $y$  are composed of various process errors (e.g., random, bias, resolution, environmental, operator, etc.).

We recall that Axiom 3 states that the uncertainty in the value of an error is equal to the square root of the variance of the error distribution. As a consequence, we can apply the **variance addition rule** to obtain a method for correctly combining uncertainties from different error sources.

### Variance Addition Rule

The variance of  $z$  can be expressed in terms of the variances of the individual variables,  $x$  and  $y$

$$\begin{aligned} \text{var}(z) &= \text{var}(ax + by) \\ &= a^2 \text{var}(x) + b^2 \text{var}(y) + 2ab \text{cov}(x,y) \end{aligned}$$

where the last term is the **covariance** between  $x$  and  $y$ . If we recall Axiom 3 and equation (9), we can express the variance of  $z$  as shown in equation (17).

$$u_z^2 = a^2 u_x^2 + b^2 u_y^2 + 2ab \text{cov}(x, y) \quad (17)$$

If two variables  $x$  and  $y$  are described by a joint probability density function  $f(x,y)$ , then the covariance of  $x$  and  $y$  is given by

$$\text{cov}(x, y) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} (x - \mu_x)(y - \mu_y) f(x, y) dy \quad (18)$$

where  $\mu_x$  and  $\mu_y$  are mean values for  $x$  and  $y$ , respectively.

The covariance is rarely used explicitly. Instead, we use the **correlation coefficient**,  $\rho_{x,y}$ , which is defined as

$$\rho_{x,y} = \frac{\text{cov}(x, y)}{u_x u_y} \quad (19)$$

With this relationship, the variance in the sum of the two quantities  $x$  and  $y$  is given by equation (20).

$$u_z^2 = a^2 u_x^2 + b^2 u_y^2 + 2ab\rho_{x,y} u_x u_y \quad (20)$$

Equation (20) can be generalized to cases where there are  $n$  measured quantities  $x_1, x_2, \dots, x_n$ .

$$\begin{aligned} \text{var}\left(\sum_{i=1}^n a_i x_i\right) &= \sum_{i=1}^n a_i^2 \text{var}(x_i) + 2\sum_{i=1}^n \sum_{j>i}^n a_i a_j \rho_{ij} u_i u_j \\ &= \sum_{i=1}^n a_i^2 u_i^2 + 2\sum_{i=1}^n \sum_{j>i}^n a_i a_j \rho_{ij} u_i u_j \end{aligned} \quad (21)$$

### Correlations Between Measurement Process Errors

With the variance addition rule, we have a logical approach for combining uncertainties that accounts for correlations between error sources. To assess the impact of the correlated errors on combined uncertainty, let us consider the measurement of a quantity  $x$  that involves two error sources  $\varepsilon_1$  and  $\varepsilon_2$ .

$$x = x_{true} + \varepsilon_1 + \varepsilon_2$$

From Axioms 1 and 3 and the variance addition rule, the uncertainty in  $x$  is obtained from

$$\begin{aligned} u_x &= \sqrt{\text{var}(x_{true} + \varepsilon_1 + \varepsilon_2)} = \sqrt{\text{var}(\varepsilon_1 + \varepsilon_2)} \\ &= \sqrt{u_1^2 + u_2^2 + 2\rho_{1,2} u_1 u_2} \end{aligned}$$

The correlation coefficient,  $\rho_{1,2}$ , for two error sources can range in value from -1 to +1. If the two error sources are statistically independent, then  $\rho_{1,2} = 0$  and  $u_x = \sqrt{u_1^2 + u_2^2}$ . Therefore, uncertainties of statistically independent error sources are combined in a root-sum-square (RSS). Conversely, if the two error sources are strongly correlated then  $\rho_{1,2} = 1$  or -1. If  $\rho_{1,2} = 1$ , then  $u_x = \sqrt{u_1^2 + u_2^2 + 2u_1 u_2} = \sqrt{(u_1 + u_2)^2} = u_1 + u_2$ . Therefore, the uncertainties are combined linearly. When two error sources are strongly correlated and compensate for one another, then  $\rho_{1,2} = -1$  and  $u_x = \sqrt{u_1^2 + u_2^2 - 2u_1 u_2} = \sqrt{(u_1 - u_2)^2} = |u_1 - u_2|$ . Therefore, the combined uncertainty is the absolute value of the difference between the individual uncertainties.

There typically aren't any correlations between measurement process errors for a given quantity. In general, it is safe to assume that there are no correlations between the following measurement process errors.

- Random Error and Parameter Bias ( $\rho_{\text{ran,bias}} = 0$ )
- Random Error and Operator Bias ( $\rho_{\text{ran,oper}} = 0$ )
- Parameter Bias and Resolution Error ( $\rho_{\text{bias,res}} = 0$ )
- Parameter Bias and Operator Bias ( $\rho_{\text{bias,oper}} = 0$ )
- Operator Bias and Environmental Factors Error ( $\rho_{\text{oper,env}} = 0$ )
- Resolution Error and Environmental Factors Error ( $\rho_{\text{res,env}} = 0$ )
- Digital Resolution Error and Operator Bias ( $\rho_{\text{dres,oper}} = 0$ )

### Accounting for Cross-Correlations

In some instances, the measurement process errors for different quantities may be correlated. For the cylinder volume example, this would occur if the same device is used to measure the cylinder length and diameter,  $L$  and  $D$ . In this case, the uncertainty in the measuring parameter bias is the same for both quantities or components.

We account for cross-correlations by developing an expression for the correlation coefficient,  $\rho_{L,D}$ , between the total uncertainties for each component,  $u_L$  and  $u_D$ , in terms of the cross-correlations,  $\rho_{L_i,D_j}$ , between the measurement process uncertainties for each component,  $u_{L_i}$  and  $u_{D_j}$ .

$$\rho_{L,D} = \frac{1}{u_L u_D} \sum_{i=1}^{n_i} \sum_{j=1}^{n_j} \rho_{L_i, D_j} u_{L_i} u_{D_j} \quad (22)$$

### Degrees of Freedom

When uncertainties are combined, we need to know the degrees of freedom for the total uncertainty. As might be expected, the degrees of freedom for a combined uncertainty estimate is not a simple sum of the degrees of freedom for each uncertainty component. The effective degrees of freedom,  $\nu_{eff}$ , for the total uncertainty,  $u_T$ , resulting from the combination of uncertainties  $u_i$  and associated degrees of freedom,  $\nu_i$ , for  $n$  error sources can be estimated via the Welch-Satterthwaite formula<sup>3</sup> given in Annex G of the GUM.

$$\nu_{eff} = \frac{u_T^4}{\sum_i \frac{u_i^4}{\nu_i}} \quad (23)$$

### Confidence Limits and Expanded Uncertainty

As previously stated, the uncertainty,  $u$ , and degrees of freedom,  $\nu$ , can be used to establish confidence limits. These are the upper and lower limits that contain the true value,  $\mu$  (estimated by the mean value  $\bar{x}$ ), with some specified confidence level or probability,  $p$ . Confidence limits are expressed as

$$\bar{x} - t_{\alpha/2, \nu} u \leq \mu \leq \bar{x} + t_{\alpha/2, \nu} u \quad (24)$$

where the multiplier is the t-statistic,  $t_{\alpha/2, \nu}$ , and  $\alpha = 1 - p$ .

The GUM defines the term **expanded uncertainty** as "the quantity defining an interval about the result of a measurement that may be expected to encompass a large fraction of the distribution of values that could reasonably be attributed to the measurand." In less obtuse language, the expanded uncertainty is basically defined as a set of limits ( $\pm L$ ) that are expected to contain the true value of the measurand. In this context, the expanded uncertainty,  $ku$ , is offered as an approximate confidence limit, in which the coverage factor,  $k$ , is used in place of the t-statistic.

$$\bar{x} - ku \leq \text{true value} \leq \bar{x} + ku \quad (25)$$

The introduction of the expanded uncertainty is confusing at best, since it is counter-intuitive to think of an uncertainty as having a range. In actual practice, the term expanded uncertainty and uncertainty are used interchangeably. This, of course, can lead to incorrect inferences and miscommunications. To mitigate this problem, the GUM also introduced the term "standard uncertainty" to help distinguish uncertainty from expanded uncertainty. Unfortunately, confusion over and misapplication of these terms persists.

### Report the Analysis Results

When reporting the results of an uncertainty analysis, Section 7 of the GUM recommends that the following information be included:

1. The estimated value of the quantity of interest (measurand) and its combined uncertainty and degrees of freedom.
2. The functional relationship between the quantity of interest and the measured components, along with the sensitivity coefficients.
3. The value of each measurement component and its combined uncertainty and degrees of freedom
4. A list of the measurement process uncertainties and associated degrees of freedom for each component, along with a description of how they were estimated.

<sup>3</sup> This formula is based on the assumption that there are no correlations between error source uncertainties.

5. A list of applicable correlation coefficients, including any cross-correlations between component uncertainties.

It is also a good practice to provide a brief description of the measurement process, including the procedures and instrumentation used, and additional data, tables and plots that help clarify the analysis results.

## **COMPARISON OF SOFTWARE FEATURES AND CAPABILITIES**

The focus of this comparison was limited to applications or tools that apply the methods contained in the latest version of the GUM [1]. Software using Monte Carlo simulation, or other alternative analysis methods, were not included in this study. The software applications and tools evaluated herein were also limited to those readily accessible via Internet download (e.g., demos or freeware) or currently available for purchase. A freeware application and commercial product, scheduled for release in early 2004, were also included. In all, two freeware applications and five commercial products with unit prices less than \$1,500 (U.S.) were evaluated.

### **Basic Software Information**

Before we delve into an assessment of the features and capabilities of each analysis software application, it is useful to first know some information about the developer, operating system requirements, pricing, software support options, etc. For simplicity, the software summary information is given in the order of the price for a single-user license.

#### **Uncertainty Calculator**

This freeware application was initially developed by Chris Grachanen of Compaq in the mid 1990's to conduct uncertainty analyses for direct measurements. It is one of the first uncertainty analysis applications offered for general use. Its freeware status has made it a widely distributed application.

Single-user Price:	Free
Developer:	Compaq
Distributor:	Agilent Technologies ( <a href="http://metrologyforum.tm.agilent.com">http://metrologyforum.tm.agilent.com</a> )
Latest Version:	3.2, released in March 2002
Version Upgrade Options:	Free download from website.
Maintenance Updates:	Software is not officially maintained by Agilent Technologies. No service updates issued since March 2002.
Compatible Operating Systems:	Windows 98, NT and 2000.
User Manual:	A 13-page document in rich text format containing brief descriptions of features and short instruction lists.
Other Supporting Documentation:	None.
Built-in Help Features:	Pop-up list of roughly 70 topics and an option to have the program automatically step through data entry screens in a set sequence.
Additional On-screen Information:	User manual file can be opened from the program once you've mapped the location of the Write.exe or Word.exe files on your computer.
Technical Support Options:	No formal support provided, but you can email Chris Grachanen at Agilent.

#### **Uncertainty SideKick**

This freeware application was developed by Integrated Sciences Group in 2005 as an introductory tool for conducting uncertainty analyses for direct measurements. Although the quality of this application is consistent with others sold commercially, it is offered as freeware to promote proper uncertainty analysis methods and procedures.

Single-user Price:	Free via download from website.
Developer:	Integrated Sciences Group (ISG)
Distributor:	Integrated Sciences Group ( <a href="http://www.isgmax.com">http://www.isgmax.com</a> )
Latest Version:	1.0 released August 2005.
Version Upgrade Options:	Free download from website.

Maintenance Updates:	Free downloads available from website.
Compatible Operating Systems:	Windows 95, 98, NT, ME, 2000 and XP
User Manual:	A 70+ page document including program features, operating procedures, uncertainty analysis basics and analysis examples in Adobe Acrobat format.
Other Supporting Documentation:	Free articles and papers available for download from website.
Built-in Help Features:	Full Help function with tree-view table of contents, indexed list, and search capabilities for 200+ topics. All analysis screens have an interactive analysis procedure checklist that provides a short, structured walk-through of basic steps in estimating uncertainty for a typical measurement.
Additional On-screen Information:	Analysis screens also contain an option for displaying detailed procedures.
Technical Support Options:	Free technical support available for registered users via phone, email or fax.

### Assistant

This software product was originally developed around 1999 by the Group for Infrared Metrology (INFRAMET) at the Institute of Optoelectronics of the Military University of Technology in Warsaw, Poland. In 2001, product maintenance and distribution responsibilities were transferred to Luxon Software.

Single-user Price:	\$149 (U.S.). Available via website download only.
Special Discount Offers:	A 50% discount is offered to scientific and educational centers.
Developer:	INFRAMET
Distributor:	Luxon Software ( <a href="http://www.luxonsoft.com">http://www.luxonsoft.com</a> )
Latest Version:	2.3 released sometime in 2003.
Version Upgrade Options:	None currently listed.
Maintenance Updates:	None currently listed.
Compatible Operating Systems:	Windows 95, 98, NT, ME, 2000 and XP
User Manual:	None.
Other Supporting Documentation:	None.
Built-in Help Features:	Full Help function with tree-view table of contents, indexed list, and search capabilities for 100+ topics.
Technical Support Options:	Questions or comments can be submitted via email.
Training Options:	None listed.

### Uncertainty SideKick Pro

This software product was developed by Integrated Sciences Group in 2005 for conducting uncertainty analyses for direct and multivariate measurements. Uncertainty SideKick Pro is designed to import analysis files from the freeware version.

Single-user Price:	\$295 (U.S.) plus shipping & handling. Available on CD only.
Special Discount Offers:	Free to U. S. college and university professors teaching related courses.
Developer:	Integrated Sciences Group
Distributor:	Integrated Sciences Group ( <a href="http://www.isgmax.com">http://www.isgmax.com</a> )
Latest Version:	1.0 released January 2006.
Version Upgrade Options:	None currently listed.
Maintenance Updates:	Free downloads available from website or \$30 (U.S) for latest program CD.
Compatible Operating Systems:	Windows 95, 98, NT, ME, 2000 and XP
User Manual:	A 140 page document including program features, operating procedures, uncertainty analysis basics and analysis examples in hardcopy format.

Other Supporting Documentation:	Free articles and papers available for download from website.
Built-in Help Features:	Full Help function with tree-view table of contents, indexed list, and search capabilities for 350+ topics. All major screens have an interactive analysis procedure checklist that provides a short, structured walk-through of basic steps in estimating uncertainty.
Additional On-screen Information:	Analysis screens also contain an option for displaying detailed procedures.
Technical Support Options:	Free technical support available for registered users via phone, email or fax.
Training Options:	Selected open-enrollment and in-house training courses offered in U.S. and Canada. Course information is listed on website.

### Uncertainty Toolbox

This Excel add-in product, developed by Quametec Corporation in 2002-03, provides built-in macros and worksheet templates for conducting uncertainty analysis for direct and multivariate measurements.

Single-user Price:	\$695 (U.S.) via website download. \$725 plus shipping & handling for CD.
Special Discount Offers:	For 2 or more users, the per unit price is \$650 (U.S.) via website download or \$675 plus shipping & handling for CD.
Developer:	Quametec Corporation
Distributor:	Quametec Corporation ( <a href="http://www.quametec.com">http://www.quametec.com</a> )
Latest Version:	2.8x, released in April 2009.
Version Upgrade Options:	Upgrade available for \$495.
Maintenance Updates:	One year of free updates via website download. Thereafter, updates can be downloaded from website for an annual "Users Group" fee of \$150 (U.S.).
Compatible Operating Systems:	All Windows operating systems capable of running MS Excel.
User Manual:	A 80 page document describing program features and operating instructions available in Adobe Acrobat format.
Other Supporting Documentation:	A Quametec book entitled "Measurement Uncertainty Analysis Fundamentals" can be purchased with software CD for an additional fee.
Built-in Help Features:	Basic Help function with table of contents for roughly 50 topics.
Additional On-screen Information:	The Master Template file contains a worksheet showing a 12-step uncertainty analysis process diagram with buttons to activate pop-up info screens. Some short-cut tips are also listed on this worksheet.
Technical Support Options:	One year of free technical support via phone or email. Thereafter, technical support is available if you pay the annual Users Group fee.
Training Options:	Selected open-enrollment and in-house training courses offered in U.S. and Canada. Course information is listed on website.

### UncertaintyAnalyzer

This software product was originally developed and released by Integrated Sciences Group in 1994 to conduct uncertainty analyses for direct measurements, multivariate measurements, and measurement systems. At that time, it was the first commercially available software product that fully implemented the methods documented in the GUM. Analysis files developed with the freeware or professional versions of Uncertainty SideKick can be imported into UncertaintyAnalyzer 3.0.

Single-user Price:	\$995 (U.S.) plus shipping & handling. Available on CD only.
Special Discount Offers:	Network license discounts are available for 6 or more users.
Developer:	Integrated Sciences Group
Distributor:	Integrated Sciences Group ( <a href="http://www.isgmax.com">http://www.isgmax.com</a> )
Latest Version:	3.0, released in November 2005.

Version Upgrade Options:	Upgrade from version 2.0 available for \$395 (U.S.). Upgrade from version 1.0 available for \$795 (U.S.).
Maintenance Updates:	Free downloads available from website or \$30 (U.S) for latest program CD. Several updates have been issued since initial product release.
Compatible Operating Systems:	Windows 95, 98, NT, ME, 2000 and XP.
User Manual:	A 260 page document including program features, operating procedures and analysis examples in hardcopy format.
Other Supporting Documentation:	Free articles and papers available for download from website.
Built-in Help Features:	Full Help function with tree-view table of contents, indexed list, and search capabilities for 650+ topics. All analysis screens have an interactive analysis procedure checklist that provides a short, structured walk-through of basic steps in estimating uncertainty for a typical measurement.
Additional On-screen Information:	Analysis screens contain option for displaying detailed procedures. The Analysis Guide screen provides a tree-view presentation of uncertainty analysis concepts and methods, nomenclature and terminology, program features and operating procedures, and analysis examples.
Technical Support Options:	Free technical support available for registered users via phone, email or fax.
Training Options:	Selected open-enrollment and in-house training courses offered in U.S. and Canada. Course information is listed on website.

### **GUM Workbench**

This software product was originally developed by Metrodata GmbH in 1997-98 to analyze uncertainties for direct and multivariate measurements to meet the requirements of document EAL-R2 of the European Cooperation for Accreditation of Laboratories.

Single-user Price:	\$1,250 (Euro) for version 1.3 plus shipping and handling. Prices are subject to change based on exchange rate of Euros to US dollars. Available on CD only.
Special Discount Offers:	None listed.
Developer:	Metrodata GmbH
Distributors:	Danish Technological Institute ( <a href="http://www.gum.dk">http://www.gum.dk</a> )
Latest Version:	1.3, released June 2005.
Version Upgrade Options:	Upgrade options offered by Danish Technological Institute only. All prices are in Euros.
Maintenance Updates:	Free updates within six months of purchase date. Must contact Metrodata GmbH for update. No maintenance updates since product release.
Compatible Operating Systems:	Windows 95, 98, NT, ME, 2000 (Version 1.2), and XP (Version 1.3).
User Manual:	A 100 page document describing program features and operating procedures is available in hardcopy or Adobe Acrobat format.
Other Supporting Documentation:	None.
Built-in Help Features:	Full Help function with table of contents, indexed list and search capabilities for roughly 100 topics.
Technical Support Options:	None listed.
Training Options:	None listed.

### **Technical Features and Capabilities**

We will now address the criteria for assessing the capabilities and features of the different software applications. These criteria focus on the uncertainty analysis steps of identifying error sources and distributions, estimating uncertainties, combining uncertainties and reporting the results.

### Overview of Main Software Features and Capabilities

When comparing the technical merits of the various software applications, it is first useful to provide an overview of the relative capabilities of each program or tool. Once this is achieved, the product features and capabilities we can be compared in more detail, as desired. The criteria for comparing the main software features and capabilities are listed in Table 1.

A ✓ in a cell indicates that the particular capability or feature is available. Alternatively, a blank cell indicates that the capability or feature is not available. Comments and footnotes are included to clarify significant differences between software applications or to provide necessary details.

**Table 1 Overview of Main Software Features and Capabilities**

Features and Capabilities	Uncertainty Calculator 3.2	Uncertainty SideKick 1.0	Assistant 2.3	Uncertainty SideKick Pro 1.0	Uncertainty Toolbox 2.8x	Uncertainty Analyzer 3.0	GUM Workbench 1.3
Analysis of Direct Measurements	✓	✓	✓	✓	✓	✓	✓
Analysis of Multivariate Measurements			✓	✓	✓	✓	✓
Analysis of Measurement Systems <sup>4</sup>						✓	
Measurement Units Database <sup>5</sup>		✓		✓		✓	
Measurement Equipment Database <sup>6</sup>				✓		✓	
Entry and Display of Analysis Information	Scratchpad/Remarks Screen	Notes Screens	Comment/Description Fields	Notes Screens	Comment/Description Fields	Notes and Analysis Description Screens	Comment/Description Fields
Bayesian Analysis		✓		✓		✓	
Uncertainty Growth Projection						✓	
Uncertainty Budget Table	✓	✓	✓	✓	✓	✓	✓
Charts and Plots	Selected Bar Charts	Distribution Plots, Pareto Charts		Distribution Plots, Pareto Charts		Distribution Plots, Histograms, Pareto Charts	

<sup>4</sup> Software provides a structured user-interface for analyzing system modules and computing uncertainty propagation from system input through system output.

<sup>5</sup> Database containing measurement areas and units consistent with NIST Special Publication 811. Editable via user interface.

<sup>6</sup> Relational database for storing and retrieving instrument identification information, tolerancing data, and other operating and manufacturing specifications.

### Identifying Error Sources and Distributions

As mentioned earlier, measurement process errors are the basic elements of an uncertainty analysis. Therefore, it is important to compare how the different software applications assist the user in identifying and describing measurement process errors. A comparison of key features and capabilities is given in Table 2. The table entry protocol is the same as that used for the previous comparison table.

**Table 2 Software Features and Capabilities for Identifying Error Sources and Distributions**

Features and Capabilities	Uncertainty Calculator 3.2	Uncertainty Sidekick 1.0	Assistant 2.3	Uncertainty Sidekick Pro 1.0	Uncertainty Toolbox 2.8x	Uncertainty Analyzer 3.0	GUM Workbench 1.3
Max. Number of Error Sources Allowed <sup>7</sup>	40	15	Over 1,000	50	25	Over 1,000	512
Identifies Measurement Process Errors <sup>8</sup>		✓		✓		✓	
Available Error Distributions <sup>9</sup>	Normal Rectangular Triangular U-Shaped Student's t	Normal Lognormal Exponential Uniform Student's t	Normal Quadratic Cosine Trapezoidal Uniform Triangular U-Shaped Student's t	Normal Lognormal Exponential Uniform Student's t	Normal Rectangular Triangular U-Shaped Student's t	Normal Lognormal Exponential Quadratic Cosine Uniform Triangular U-Shaped Student's t	Normal Trapezoidal Rectangular Triangular U-Shaped Student's t
Error Distribution Plots		✓		✓		✓	
Default Distribution Selection	Student's t with Degrees of Freedom >100	See Footnote <sup>10</sup>		See Footnote <sup>10</sup>		See Footnote <sup>10</sup>	Rectangular
Assistance in Selecting Appropriate Distributions						See Footnote <sup>11</sup>	
Computation of Bounding Limits for Distributions						✓	

<sup>7</sup> All measurement process errors that can be defined and analyzed in templates, screens and worksheets.

<sup>8</sup> Interactive procedure checklists and error source tables automatically include most commonly encountered measurement process errors such as operator bias, environmental factors error, and resolution error. User defined error sources can also be included.

<sup>9</sup> The uniform and rectangular distributions are the same.

<sup>10</sup> If no distribution is selected, the program selects an appropriate one based on user specified error containment or tolerance limits and confidence level (%) or coverage factor.

<sup>11</sup> Descriptions and applications of various error distributions are provided in the on-screen Help function. Detailed selection guidelines are also given in UncertaintyAnalyzer's Analysis Guide screen under the topic "Choosing the Appropriate Error Distribution."

### Estimating Uncertainties

The next step in the analysis procedure is to estimate the uncertainties in the measurement process errors. As was discussed earlier, there are two ways to estimate uncertainties. One is based on the analysis of sampled data (Type A estimates) and the other is based on a heuristic approach (Type B estimates). Given the marked difference in these approaches, it is best to separately evaluate the software capabilities for each type of uncertainty estimate. Comparisons of software features and capabilities for conducting Type A and Type B uncertainty estimates are given in Tables 3 and 4, respectively.

**Table 3 Software Features and Capabilities for Conducting Type A Uncertainty Estimates**

Features and Capabilities	Uncertainty Calculator 3.2	Uncertainty SideKick 1.0	Assistant 2.3	Uncertainty SideKick Pro 1.0	Uncertainty Toolbox 2.8x	Uncertainty Analyzer 3.0	GUM Workbench 1.3
Data Analysis Options	Sampled Values	Sampled Values or Sampled Mean Values	Sampled Values	Sampled Values or Sampled Mean Values	Sampled Values	Sampled Values, Sampled Mean Values, Sampled Cells, or Mixed <sup>12</sup>	Sampled Values
Maximum Sample Size	50	200	> 1,000	200	30	2,000	9,999
Data Import Options	Formatted Text File	Paste from Clipboard	Formatted Text File or Paste from Clipboard	Delimited Text, Excel, Lotus, Various Database Formats, or Paste from Clipboard	Paste from Clipboard	Delimited Text, Excel, Lotus, Various Database Formats, or Paste from Clipboard	Formatted Text, Excel, or Paste from Clipboard
Data Export Options	Formatted Text File	Paste to Clipboard	Paste to Clipboard or Formatted Text File	Paste to Clipboard	Paste to Clipboard	Paste to Clipboard	Tab Delimited Text, Rich Text, or HTML
Computed Sample Statistics	Min., Max., and Mean Values, Standard Deviation, Variance	Mean Value, Standard Deviation, Standard Deviation of Mean Value	Mean Value, Standard Deviation	Mean Value, Standard Deviation, Standard Deviation of Mean Value	Mean Value, Standard Deviation, Standard Deviation of Mean Value	Mean Value, Standard Deviation, Standard Deviation of Mean Value, Skewness, Kurtosis	Mean Value, Standard Deviation, Standard Deviation of Mean Value
Data Correction Options		See Footnote <sup>13</sup>		See Footnote <sup>13</sup>	See Footnote <sup>14</sup>	See Footnote <sup>13</sup>	
Advanced Statistical Analysis Tools						See Footnote <sup>15</sup>	

<sup>12</sup> The mixed option allows for use of a mean value computed from sample values combined with a user specified standard deviation and sample size obtained from a prior sample of data that characterizes the repeatability uncertainty of a measurement process.

<sup>13</sup> Corrections for parameter bias and/or environmental factors can be applied to the sample mean by selecting check boxes.

<sup>14</sup> Correction for thermal expansion is computed, but user must manually apply this value to sample mean.

<sup>15</sup> Built-in analysis tools for computing statistics for a sample subset, normality testing, and outlier identification and removal.

**Table 4 Software Features and Capabilities for Conducting Type B Uncertainty Estimates**

Features and Capabilities	Uncertainty Calculator 3.2	Uncertainty SideKick 1.0	Assistant 2.3	Uncertainty SideKick Pro 1.0	Uncertainty Toolbox 2.8x	Uncertainty Analyzer 3.0	GUM Workbench 1.3
Analysis Options for Estimating Type B Uncertainty	Enter Standard Uncertainty for Selected Distribution	Enter Standard Uncertainty for Selected Distribution	Enter Standard Uncertainty for Student's t Distribution	Enter Standard Uncertainty for Selected Distribution		Enter Standard Uncertainty for Selected Distribution	Enter Standard Uncertainty for Student's t Distribution
	Enter Expanded Uncertainty & Coverage Factor for Selected Distribution	Enter Containment Limits <sup>16</sup> & Coverage Factor for Selected Distribution	Enter Expanded Uncertainty & Coverage Factor for Normal Distribution	Enter Containment Limits <sup>16</sup> & Coverage Factor for Selected Distribution	Enter Uncertainty Limits & Confidence Level for Normal Distribution	Enter Containment Limits <sup>16</sup> & Coverage Factor for Selected Distribution	Enter Expanded Uncertainty & Coverage Factor for Normal Distribution
	Enter Expanded Uncertainty & Confidence Level for Selected Distribution	Enter Containment Limits & Confidence Level for Selected Distribution	Enter Bounding Limits for Other Distributions	Enter Containment Limits & Confidence Level for Selected Distribution	Enter Uncertainty Limits for Other Distributions	Enter Containment Limits & Confidence Level for Selected Distribution	Enter Bounding Limits for Other Distributions
Advanced Screens for Type B Estimates		See Footnote <sup>17</sup>		See Footnote <sup>17</sup>		See Footnote <sup>17</sup>	
Symmetric & Asymmetric Two-sided Tolerance or Error Limits		✓	✓	✓		✓	
One-sided Tolerance or Error Limits		✓		✓		✓	
Compute Tol. Limits from Linear or RSS Combination of Instrument Specifications		✓		✓	✓	✓	
Advanced Analysis Tools						See Footnote <sup>18</sup>	
Degrees of Freedom Estimation	See Footnote <sup>19</sup>	See Footnote <sup>20</sup>	See Footnote <sup>19</sup>	See Footnote <sup>20</sup>	See Footnote <sup>19</sup>	See Footnote <sup>20</sup>	

<sup>16</sup> Containment limits refer to expanded uncertainty, tolerance limits, or other error limits.

<sup>17</sup> Specialized drill-down screens for estimating uncertainties due to resolution error, operator bias, environmental factors error, etc.

<sup>18</sup> Built-in SpecMaster tool for computing tolerance limits from complex specifications using VB Script programming language.

<sup>19</sup> Simple calculation based on Equation G.3 of the GUM [1]. User is required to enter "relative" uncertainty in the uncertainty.

<sup>20</sup> Built-in Type B Degrees of Freedom Calculator, that also uses Equation G.3, but provides a user-friendly, yet rigorous approach to simultaneously estimating uncertainty and degrees of freedom [5].

### Combining Uncertainties

The next step in the analysis procedure is to combine the uncertainty estimates for the measurement process errors. This step requires that we account for any correlations between measurement process errors and, for multivariate analyses, any cross-correlations between measurement components. For multivariate analyses, it is also important that the uncertainties are weighted or multiplied by the appropriate sensitivity coefficients. Lastly, we must also estimate the effective degrees of freedom associated with the combined uncertainty. A comparison of key software features and capabilities is given in Table 5.

**Table 5 Software Features and Capabilities for Combining Uncertainties**

Features and Capabilities	Uncertainty Calculator 3.2	Uncertainty SideKick 1.0	Assistant 2.3	Uncertainty SideKick Pro 1.0	Uncertainty Toolbox 2.8x	Uncertainty Analyzer 3.0	GUM Workbench 1.3
Correlations Between Measurement Process Errors	Limited to One Pair of Error Sources	✓	✓	✓	Limited to Two Pairs of Error Sources	✓	✓
Cross-correlations Between Component Errors			✓	✓	Limited to Two Pairs of Error Sources	✓	✓
Correlation Coefficients Table		✓	✓	✓		✓	✓
Automatic Computation of Sensitivity Coefficients <sup>21</sup>			✓	✓	See Footnote <sup>22</sup>	✓	✓
User Defined Sensitivity Coefficient Equations						✓	
Effective Degrees of Freedom <sup>23</sup>		✓	✓	✓	✓	✓	✓
Computation of Confidence Limits or Expanded Uncertainty <sup>24</sup>	✓	✓	✓	✓	✓	✓	✓

<sup>21</sup> In a multivariate analysis, the sensitivity coefficient of a variable is the partial derivative of the equation that defines a quantity of interest with respect to that variable. The product of the sensitivity coefficient and the variable's standard uncertainty reflects the contribution of the uncertainty in the variable's error to the overall uncertainty in the computed quantity.

<sup>22</sup> A Sensitivity Coefficient Worksheet can be used to compute partial derivatives for up to 10 variables. The computed sensitivity coefficients must then be copied and pasted into the appropriate cells on the Uncertainty Budget Worksheet.

<sup>23</sup> Computed using Welch-Satterthwaite formula given in Annex G of the GUM [1].

<sup>24</sup> Computed by multiplying combined uncertainty by 1) user specified coverage factor or 2) coverage factor computed from effective degrees of freedom and user specified confidence level. Normal distribution is used for infinite degrees of freedom, otherwise Student's t distribution is used.

### Reporting the Analysis Results

The last step in the uncertainty analysis procedure is to report the results. In some instances, the uncertainty analysis report will be a key document for compliance with ISO/IEC 17025 or other certification requirements. Therefore, it is useful to compare the reporting features and capabilities of the different software applications. A comparison of key reporting features and capabilities is given in Table 6.

**Table 6 Software Features and Capabilities for Reporting Uncertainty Analysis Results**

Features and Capabilities	Uncertainty Calculator 3.2	Uncertainty SideKick 1.0	Assistant 2.3	Uncertainty SideKick Pro 1.0	Uncertainty Toolbox 2.8x	Uncertainty Analyzer 3.0	GUM Workbench 1.3
Summary Report <sup>25</sup>	✓	✓	✓	✓	✓	✓	✓
Drill-down Reports <sup>26</sup>		✓		✓	✓	✓	✓
Report Customization Options <sup>27</sup>		✓		✓	✓	✓	
Print Preview Option		✓	✓	✓	✓	✓	✓
Export Options	Analysis Reports	Notes, Charts & Plots, Analysis Reports	Report & Selected Portions of Analysis Screens	Notes, Charts & Plots, Analysis Reports	Excel Copy/Paste & File Save Options	Notes, Analysis Description, Charts & Plots, Reports	Report & Selected Portions of Analysis Screens
Export Formats	Microsoft Write, Word and Excel File Formats	Windows Clipboard, Rich Text, and HTML File Formats	Windows Clipboard, Excel, Rich Text, HTML File Formats	Windows Clipboard, Rich Text and HTML File Formats	Standard Excel File Save Options	Windows Clipboard, Rich Text, and HTML File Formats	Windows Clipboard, Tab Delimited Text, Rich Text and HTML File Formats

### RECOMMENDED GUIDELINES FOR SELECTING ANALYSIS SOFTWARE

Price is always a key factor in the selection of an uncertainty analysis software application. However, there are several additional factors that should be considered.

- User Interface
- Built-in Help Features
- User Manual and other Supporting Documentation
- Technical Support Options
- Training Options
- Network Installation and Site License Options
- Software Maintenance
- Product Upgrades
- Software Validation

<sup>25</sup> An analysis report containing user specified title and other administrative information, basic analysis description, uncertainty budget table, combined uncertainty and effective degrees of freedom.

<sup>26</sup> Detailed Type A or Type B uncertainty analysis reports for each measurement process error.

<sup>27</sup> Options for changing report fonts, selecting which reports to print, including or excluding notes or comments, and including or excluding charts, plots and other graphics (if available).

These factors primarily address the professional capabilities and qualifications of software developers and distributors. While some of the selection criteria were included in the software comparison section of this paper, it is important to discuss them in more detail.

### **User Interface**

A primary factor in the selection of an uncertainty analysis program should be how well the user can access the full functionality of the program. This is primarily done via a graphical user interface consisting of screens, templates, or spreadsheets. The design, layout and organization of the screens, menus and toolbars should facilitate program use. They should also promote an understanding of how the application works.

Because most users will not have advanced training in uncertainty analysis methods, program screens should be designed to provide technical guidance through the inclusion of interactive procedures or step-by-step checklists. To eliminate confusion, clear and consistent terminology should be used throughout the program.

The program should also include error traps to ensure that realistic information and data are entered in the appropriate fields and cells. Error trapping is more difficult to achieve with Excel plug-in applications because the cells in which data are entered cannot be completely secure. In many instances, hidden and locked spreadsheet cells are difficult to secure without significantly reducing user interaction.

### **Built-in Help Features**

Another important aspect of an analysis program is the inclusion of a comprehensive Help file that is easily accessed from all screens, templates or worksheets. The contents of the Help file should be presented in a tree-view format for easy navigation. The Help file should also have complete index and search capabilities of all topics to facilitate program use. All Help topics should be written in a concise manner that clearly conveys the appropriate information.

Additional on-screen help features should include strategically positioned context sensitive information buttons that provide convenient, short cut access to corresponding Help file topics. As previously mentioned, the inclusion of interactive procedures and step-by-step checklists also provide valuable assistance when using a program.

### **User Manual and Other Supporting Documentation**

Most commercial software applications ship with a user manual that covers program installation, features, and operating instructions. The user manual should include a table of contents and index of topics to facilitate quick access to desired information. The user manual can be in hard copy or in selected file formats such as Adobe Acrobat pdf or Word.

Additional documentation, covering software update or upgrade features and operating instructions should also be readily available in hardcopy or pdf file formats. Articles and papers covering the methods and techniques incorporated into an analysis program are a very useful resource. While it is standard practice to list technical references for a software application, it is preferable to have easy access to original documents written by the developer.

### **Technical Support Options**

Technical support is a major consideration for most software users, especially when using a specialized analysis program. Some software companies may provide unlimited support at no additional fee, while others may require a service fee. Options for contacting technical support personnel (e.g., phone, fax, email, ...) should also be considered.

In many cases, customers may require guidance in how to conduct an uncertainty analysis. Therefore, it is important to determine if technical support is limited to application use only. If not, then the expertise of the technical support personnel should be a factor when selecting an analysis program. Technical qualifications and credentials should be readily available from the software website.

### **Training Options**

Some companies or user organizations may require or desire formal training in the use of an analysis application. In this case, it is important to review the training options provided by the software developer or supplier. Standard training options should include selected off-site, open enrollment courses; on-site or in-house courses; and customized one-on-one training. Course descriptions, outlines, pricing and instructor information should be readily available via the software website.

### **Network Installation and Site License Options**

Some companies or user organizations may require or desire that the software application be installed on a network server to allow multi-user access. In this case, the software application must be network configurable and provide a multi-user, site license pricing option. Networking capabilities and site licensing options should be readily available via the software website.

### **Software Maintenance**

Software maintenance and support should be a major consideration when comparing analysis applications. The level of software maintenance and support provided by an application developer is a direct indication of their commitment to product excellence.

Software maintenance or service updates primarily focus on bug fixes, but some companies also include product enhancements as well. While it is customary to charge for major software upgrades, maintenance or service updates are typically free of charge and readily available via download from the software website. Supporting documentation covering program fixes, changes and modifications should also accompany software updates.

### **Product Upgrades**

Long-term commitment to software support is reflected by the ability or inability of a developer to provide timely product upgrades. In general, software upgrades address new computer operating system requirements, include major enhancements and improved capabilities, and incorporate new analysis methods and techniques.

Software upgrades should provide a significant improvement or change to the product and be offered at a price that is commensurate with the value added. An updated user manual or other supporting documentation should accompany the product upgrade.

### **Software Validation**

Currently, there are no standards or guidelines for testing and validating uncertainty analysis software. However, there are many common-sense protocols, some of which are listed below.

- Validation of mathematical and statistical methods.
- Verification of numerical approximations and calculations.
- Verification of program functionality.

### **Mathematical and Statistical Methods**

In general, it is not sufficient to simply state that an uncertainty analysis application incorporates the methods described in the GUM [1]. The software developer should also publish papers and articles that clearly describe the mathematical and statistical concepts that are incorporated in the product. This serves two purposes:

- 1) It shows whether or not the developer has a sufficient technical understanding of uncertainty analysis concepts and principles.
- 2) The information can be reviewed and scrutinized in the public domain.

### **Numerical Approximations and Calculations**

Depending upon the sophistication of the numerical algorithms, program calculations can be verified via hand calculations, Excel spreadsheets, or math and statistics applications such as MathCAD or Mathematica. Verification of numerical algorithms can be achieved in a number of ways including:

1. Alpha testing via internal peer review and verification.
2. Beta testing via external review and verification by selected customer base.
3. Widespread peer review and verification via distribution of freeware subprograms and applets.
4. Large-scale customer use and feedback. This applies to well established analysis programs that have been used by 100+ customers for the past several years or more.

None of the uncertainty analysis products reviewed in this paper currently have official software validation documentation. However, some software developers do provide customers with specific validation reports upon request. If software validation is of primary concern, then it is advisable that prospective purchasers request a brief description of the developer's software validation protocol.

### **Program Functionality**

Another important aspect of software validation is the verification that the program screens, templates, or worksheets function as intended. For example, data entered into a drill-down screen should be properly stored and transferred to other screens as needed. The protocol for testing and validating program functionality is the same as described for numerical algorithms.

## **NOMENCLATURE**

This paper employs terms and definitions that are designed to be understood across a broad technology base. For further clarification, a list of terms and corresponding definitions is included.

<b>Name</b>	<b>Definition</b>
Artifact	A measurable physical object or substance.
Asymmetric Tolerances	Two-sided tolerance limits in which the magnitudes of the upper and lower limits are unequal.
Attribute	The characteristic, feature, or aspect of an object or substance.
Bias	A systematic discrepancy between an indicated or declared value of an artifact and its true value.
Bias Uncertainty	The uncertainty in the bias of a parameter or artifact.
Combined Uncertainty	The uncertainty in the total error of the value of a quantity of interest.
Component Error	The error in the measurement of a given component of a multivariate measurement.
Computation Error	The error in a quantity obtained by computation.
Computed Value or Quantity	The value computed from other measured quantities or variables via a mathematical equation defining the relationship between the computed quantity and the other variables.
Confidence Level	The probability that a set of error limits or containment limits will contain errors for a given error source.
Confidence Limits	Limits that bound errors for a given error source with a specified probability or confidence.
Containment Limits	Limits that bound errors for a given error source with a specified probability.
Containment Probability	The probability that measurement errors for a given source will lie within specified containment limits.
Correlation Coefficient	A measure of the extent to which two error sources are linearly related.
Covariance	The covariance of two error sources is a measure of their statistical dependence.
Coverage Factor	A multiplier used to express an error limit or expanded uncertainty as a multiple of the standard uncertainty.
Cross-correlation	The correlation between two error sources for two different components of a multivariate analysis.
Degrees of Freedom	A statistical quantity that is related to the amount of information available about an uncertainty estimate. It also serves as a useful statistic in determining appropriate coverage factors and computing confidence limits and other decision variables.

Direct Measurements	Measurements in which a measuring parameter X directly measures the value of a subject parameter Y (i.e., X measures Y).
Effective Degrees of Freedom	The degrees of freedom for combined uncertainties computed from the Welch-Satterthwaite formula.
Error Distribution	A probability distribution that describes the relative frequency of occurrence of values of a measurement error.
Error Limits	Bounding values that are expected to contain the error from a given source with some specified level of probability or confidence.
Error Source	A parameter, variable or constant that can contribute error to the determination of the value of a quantity of interest.
Error Source Uncertainty	The uncertainty in the error of a given source.
Expanded Uncertainty	A multiple of the standard uncertainty reflecting either a specified confidence level or arbitrary coverage factor.
Heuristic Estimate	An estimate resulting from accumulated experience and/or technical knowledge concerning the uncertainty of an error source.
Histogram	See Sample Histogram.
Independent Error Sources	Error sources that are statistically independent.
Interaction Coefficient	A coefficient that relates an environment/ancillary factor to an error source.
Mean Value	<i>Sample Mean</i> : The average value of a measurement sample. <i>Population Mean</i> : The expectation value for measurements sampled from a population.
Measurand	According to Annex B, Section B.2.9 of the GUM, the measurand is defined as "the particular quantity subject to measurement."
Measurement Area	A measurement discipline, such as DC Voltage, Temperature, or Mass.
Measurement Error	The difference between the measured value of a quantity and its true value.
Measurement Process Errors	Errors resulting from the measurement process (e.g., measuring parameter bias, operator bias, environmental factors, resolution,...).
Measurement Process Uncertainties	See Error Source Uncertainty
Measurement Uncertainty	The uncertainty in a measurement or in the error in the measurement.
Measurement Units	The units, such as volts, millivolts, etc., in which a measurement or measurement error is expressed.
Measuring Parameter	Attribute of a measuring device that is used to obtain information that quantifies the value of the subject parameter.
Mode Value	The value of a parameter or quantity most often encounter or measured.
Multivariate Measurements	Measurements in which the subject parameter or quantity of interest is computed from measurements of two or more variables or other quantities.
Operator Bias	Error due to the perception or influence of a human operator or other agency.
Parameter	Often thought of as a specified aspect or feature of an instrument or item. See also Attribute.
Parameter Bias	A systematic deviation of a parameter value from its nominal or indicated value.

Pareto Diagram	A bar chart that ranks the relative contribution of component uncertainties to the total combined uncertainty.
Population	The total set of possible values for a random variable under consideration.
Precision	Precision corresponds to how many places past the decimal point we can express a measurement result.
Probability	The likelihood of the occurrence of a specific event or value from a population of events or values.
Probability Density Function	A mathematical function that describes the relative frequency of occurrence of the values of a random variable.
Random Error	An error that appears as differences in the measured values of a given artifact or parameter during a measurement session.
Reference Standard	An artifact used as a measurement reference whose value and uncertainty have been determined by calibration.
Reliability Model	A mathematical function whose characteristics are based on a parameter's calibration history. Used to project uncertainty growth over time.
Repeatability	The closeness of the agreement between the results of successive measurements of the value of a parameter carried out under the same measurement conditions.
Resolution Error	The error due to the finiteness of the precision of a measurement.
RSS	Root-sum-square method of combining values.
Sample	A collection of values drawn from a population.
Sample Histogram	A bar chart showing the relative frequency of occurrence of sampled data.
Sample Mean	The arithmetic average of the measurements of a sample.
Sample Size	The number of measured values that comprise a sample.
Sensitivity Coefficient	For a multivariate analysis, the sensitivity coefficient for a given variable is the partial derivative of the computed value equation with respect to the variable.
Sensitivity Coefficient Equation	An equation that expresses the partial derivative of the computed value equation with respect to a given variable.
Single-sided Tolerance Limit	A tolerance limit in which only an lower or upper limit is specified.
Standard Deviation	The square root of the variance of a sample or population of values.
Standard Uncertainty	A statistic representing the spread or uncertainty in the value of a parameter or error source. If determined statistically from sampled data, it is equal to the sample standard deviation.
Statistical Independence	A property that describes two error sources as being uncorrelated.
Subject Parameter	An attribute whose value we seek to obtain from a measurement or set of measurements. Also referred to as the "measurand."
Symmetric Tolerances	Two-sided tolerance limits in which the magnitudes of the upper and lower limits are equal.
Tolerance Limits	Limits that bound acceptable parameter values.
True Value	The value that would be obtained by a perfect measurement. True values are by nature indeterminate.
Type A Estimates	Uncertainty estimates obtained by the statistical analysis of a sample of data.
Type B Estimates	Uncertainty estimates obtained by heuristic means. See also Heuristic Estimate.

Uncertainty	See Standard Uncertainty.
Uncertainty Budget	An itemized list of error sources, their associated uncertainties and degrees of freedom, and method of estimation (Type A or Type B).
Uncertainty Growth	The increase in the uncertainty in the value of a parameter or other attribute over the time elapsed since measurement.
Uncertainty in the Mean Value	Equal to the standard deviation in the sample mean divided by the square root of the sample size. Also referred to as the sampling distribution.
Variance	A measure of the spread of a distribution about its mean or mode value. Also referred to as the mean square error.

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