SELECTING AND APPLYING ERROR DISTRIBUTIONS IN UNCERTAINTY ANALYSIS[1]

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ABSTRACT

Errors or biases from a number of sources may be encountered in performing a measurement. Such sources include random error, measuring parameter bias, measuring parameter resolution, operator bias, environmental factors, etc. Uncertainties due to these errors are estimated either by computing a Type A standard deviation from a sample of measurements or by forming a Type B estimate based on experience.

In this paper, guidelines are given for selecting and applying various statistical error distribution that have been found useful for Type A and Type B analyses. Both traditional and Monte Carlo methods are outlined. Also discussed is the estimation of the degrees of freedom for both Type A or Type B estimates.

Where appropriate, concepts will be illustrated using spreadsheets, freeware and commercially available software.

INTRODUCTION

THE GENERAL ERROR MODEL

The measurement error model employed in this paper is given in the expression

$$x = x_{true} + \varepsilon \tag{1}$$

where

x	=	a value obtained by measurement
x_{true}	=	the true value of the measured quantity
ε	=	the measurement error.

With univariate direct measurements, the variable ε is composed of a linear sum of errors from various measurement error sources. With multivariate measurements ε consists of a weighted sum of error components, each of which is make up of errors from measurement error sources. More will be said on this later.

APPLICABLE AXIOMS

In order to use Eq. (1) to estimate measurement uncertainty, we invoke three axioms [HC95a]:

> Axiom 1
> Measurement errors are random variables that follow statistical distributions.

Axiom 1 provides the foundation from which to construct an uncertainty estimate. That is, although we will not ordinarily know the error in a measurement ε, we can describe it statistically.

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[1] Presented at the Measurement Science Conference, Anaheim, 2004

Axiom 2

The standard uncertainty in a measurement result is the standard deviation of the measurement error distribution.

Axiom 2 identifies the statistic of the error distribution that corresponds to the uncertainty in the measurement error ε.

Axiom 3

The uncertainty in a measurement is equal to the uncertainty in the measurement error.

Axiom 3 links the uncertainty in the measurement error ε to the uncertainty in x.

STATISTICAL DISTRIBUTIONS

A statistical distribution is a relationship between the value of a variable and its probability of occurrence. Such distributions may be characterized by different degrees of spreading or may even exhibit different shapes. A distribution may be continuous or discrete or a mixture of the two.

A variables whose probability of occurrence is described by a statistical distribution is referred to as a *random variable*.

Statistical distributions are usually expressed as a mathematical function called the *probability density function*, or *pdf*.

ERRORS AND DISTRIBUTIONS

Axiom 1 tells us that Measurement errors are random variables that follow statistical distributions. For certain kinds of error, such as repeatability or "random error," the validity of this assertion is easily seen. Conversely, for other kinds of error, such as parameter bias and operator bias, the validity of the assertion may not be so readily apparent.

What we need to bear in mind, however, is that, while a particular error may have a systematic value that persists from measurement to measurement, it nevertheless comes from some distribution of like errors that can be described statistically, i.e., that possess a probability of occurrence [HC2001].

The upshot is that, whether a particular error is random or systematic, it can still be regarded as coming from a distribution of errors that can be described statistically.

ERROR AND UNCERTAINTY

In estimating measurement uncertainty, we usually assume a zero mean or mode value for error distributions. For such cases, it is easy to grasp that, if measurement errors are tightly constrained around zero, the uncertainty in their values is small in comparison with errors that are widely spread. In other words, the spread in an error distribution is synonymous with the uncertainty in the error. This spread is quantified by the standard deviation of the distribution. Hence, Axiom 2. [2]

The standard deviation of a measurement error distribution is the square root of the distribution's *statistical variance*. Simply put, this variance is the distribution's *mean square error*. If $f(\varepsilon)$ represents the

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[2] In cases where an error distribution possesses a "known" nonzero systematic bias, the mean or mode value may not be zero. In these cases, the uncertainty estimate includes both the uncertainty in the bias and the uncertainty due to the spread of the measurement error.

probability density function for a measurement error distribution, and μ_ε represents the distribution's mode or mean value (usually zero), then the variance or mean square error is given by[3]

$$\text{var}(\varepsilon) = \int_{-\infty}^{\infty} f(\varepsilon)(\varepsilon - \mu_\varepsilon)^2 d\varepsilon \ , \tag{2}$$

and the standard deviation becomes

$$u_\varepsilon = \sqrt{\text{var}(\varepsilon)} \ . \tag{3}$$

Eqs. (2) and (3) show that the uncertainty in a measurement error can be estimated if the distribution's pdf is known. In this paper, several such pdfs will be discussed. Guidelines for selecting the applicable pdf in an uncertainty analysis will also be given.

MEASUREMENT ERROR DISTRIBUTIONS

Any one of a variety of distributions may be assumed to represent the underlying distribution of a measurement error. In this paper, we consider the uniform, normal, lognormal, quadratic, cosine, half-cosine, U-shaped, and the Student's t distribution. We also discuss the trapezoidal and utility distributions and distributions that emerge from combining errors from different sources.

THE NORMAL DISTRIBUTION

The normal distribution is the "workhorse" of statistics and probability. It is usually assumed to be the underlying distribution for random variables. Indeed, the various tools we use in applying uncertainty estimates are nearly always based on the assumption that measurement errors are normally distributed, regardless of the distributions used to estimate the uncertainties themselves.[4]

The pdf for the normal distribution is given by

$$f(\varepsilon) = \frac{1}{\sqrt{2\pi}u_\varepsilon} e^{-(\varepsilon-\mu_\varepsilon)^2/2u_\varepsilon^2} \ . \tag{4}$$

Applicability of the Normal Distribution

Why do we usually assume a normal distribution? The primary reason is because this is the distribution that either represents or approximates what we frequently see in the physical universe. It can be derived from the laws of physics for such phenomena as the diffusion of gases and is applicable to instrument parameters subject to random stresses of usage and handling. It is also often applicable to equipment parameters emerging from manufacturing processes.

An additional consideration applies to the distribution we should assume for a total error or deviation that is composed of constituent errors or deviations. There is a theorem called the *central limit theorem* that demonstrates that, even though the individual constituent errors or deviations may not be normally distributed, the combined error or deviation is approximately so. We will return to this later.

An argument has been presented against the use of the normal distribution in cases where the variable of interest is restricted, i.e., where values of the variable are said to be bound by finite physical limits. This condition notwithstanding, the normal distribution is still widely applicable in that, for many such cases, the physical limits are located far from the population mean.

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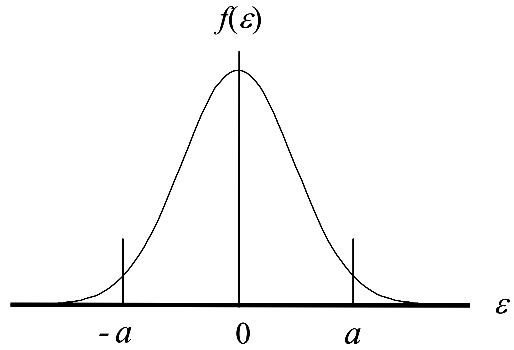
[3] Strictly speaking, Eq. (2) applies only to continuous distributions. Extending the concept to discrete distributions is straightforward.

[4] This is especially true in the GUM [ISO1997], although mention is made of other distributions and other distributions are applied in GUM examples.



The Normal Distribution. Shown is a case where the error distribution mean value is zero. The limits ±*a* are approximate 95% confidence limits.

In cases where this is not so, other feasible distributions, such as the lognormal, quadratic, cosine or half-cosine distribution can be applied.

Type A Uncertainty Estimates

When obtaining a Type A uncertainty estimate, we compute a standard deviation from a sample of values. For example, we estimate repeatability uncertainty by computing the standard deviation for a sample of repeated measurements of a given value. We also obtain a sample size. The sample standard deviation, equated with the random uncertainty of the sample, is an estimate of the standard deviation for the population from which the sample was drawn. Except in rare cases, we assume that this population follows the normal distribution.

This assumption, allows us to easily obtain the degrees of freedom and the sample standard deviation and to construct confidence limits, perform statistical tests, estimate measurement decision risk and to rigorously combine the random uncertainty estimate with other Type A uncertainty estimates.

Type B Uncertainty Estimates

A Type B uncertainty estimate is obtained using error containment limits and a containment probability.[5] The use of the normal distribution is appropriate in cases where the above considerations apply and the limits and probability are at least approximately known.

The extent to which this knowledge is approximate determines the degrees of freedom of the uncertainty estimate [HC00, ISG00]. The degrees of freedom and the uncertainty estimate can be used in conjunction with the Student's t distribution (see below) to compute confidence limits.

Let ±*a* represent the known containment limits and let *p* represent the containment probability. Then an estimate of the standard deviation of the measurement error distribution is obtained from

$$u_\varepsilon = \frac{a}{\Phi^{-1}\left(\frac{1+p}{2}\right)}, \tag{5}$$

where $\Phi^{-1}(\cdot)$ is the inverse normal distribution function. This function can be found in statistics texts and in popular spreadsheet programs.

If only a single containment limit is applicable, such as with single-sided tolerances, the appropriate expression is

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[5] The containment probability and containment limits for Type B estimates are analogous to the confidence level and confidence limits for Type A estimates.

$$u_\varepsilon = \frac{a}{\Phi^{-1}(p)} . \tag{6}$$

The degrees of freedom can be estimated for a normally distributed error according to [ISO97]

$$\nu_e \cong \frac{1}{2} \frac{u_e^2}{\sigma^2(u_e)} , \tag{7}$$

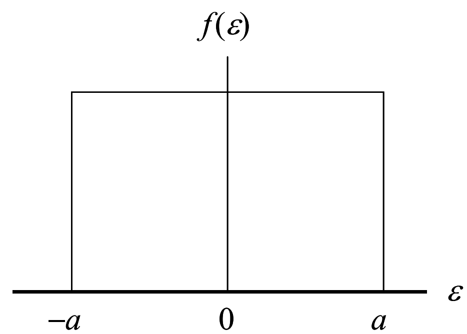
where $\sigma^2(u_\varepsilon)$ is the variance in u_ε. A methodology for estimating this variance has been developed [HC00] and has been implemented in commercially available software [ISG04] and in a freeware application [ISG03b].

THE UNIFORM OR "RECTANGULAR" DISTRIBUTION

The uniform distribution has become popular in recent years for reasons discussed below. It is defined by the probability density function

$$f(\varepsilon) = \begin{cases} \dfrac{1}{2a}, & -a \le \varepsilon \le a \\ 0, & \text{otherwise} , \end{cases} \tag{8}$$

where $\pm a$ are the limits of the distribution.



The Uniform Distribution. The probability of lying between -*a* and *a* is constant. The probability of lying outside ±*a* is zero.

Type B Uncertainty Estimates

A Type B uncertainty estimate is usually obtained using error containment limits only. When this is the case, we assume known containment limits $\pm a$ and obtain the standard deviation of the error distribution using the expression

$$u_\varepsilon = \frac{a}{\sqrt{3}} . \tag{9}$$

For this expression to be useful, the limits $\pm a$ must be *minimum containment limits* (see below). In practice, we almost never know the value of a, and should attempt to obtain a containment probability p and containment limits $\pm L$ from whence we get

$$a = \frac{L}{p}, \quad L \le a . \tag{10}$$

Once we have a value for a, the standard deviation u_ε is computed using Eq. (9).

Criteria for Applying the Uniform Distribution

The use of the uniform distribution is appropriate under a limited set of conditions. These conditions are summarized by the following criteria.

The first criterion is that we must know a set of *minimum* bounding limits for the distribution. This is the *minimum limits criterion*. Second, we must be able to assert that the probability of finding values between these limits is unity. This is the *100% containment criterion*. Third, we must be able to demonstrate that the probability of obtaining values between the minimum bounding limits is uniform. This is the *uniform probability criterion*.

Minimum Limits Criterion

It is vital that the limits we establish for the uniform distribution are the minimum bounding limits. For instance, if the limits $\pm L$ bound the variable of interest, then so do the limits $\pm 2L$, $\pm 3L$, and so on. Since the uncertainty estimate for the uniform distribution is obtained by dividing the bounding limit by the square root of three, using a value for the limit that is not the minimum bounding value will obviously result in an invalid uncertainty estimate.

This alone makes the application of the uniform distribution questionable in estimating bias uncertainty from such quantities as tolerance limits, for instance. It may be that out-of-tolerances have never been observed for a particular parameter (100% containment), but it is unknown whether the tolerances are minimum bounding limits. Some years ago, a study was conducted involving a voltage reference that showed that values for one parameter were normally distributed with a standard deviation that was approximately 1/10 of the tolerance limit. With 10-sigma limits, it is unlikely that any out-of-tolerances would be observed. However, if the uniform distribution were used to estimate the bias uncertainty for this item, based on tolerance limits, the uncertainty estimate would be nearly six times larger than would be appropriate. Some might claim that this is acceptable, since the estimate can be considered a conservative one. That may be. However, it is also a useless estimate. This point will be elaborated later.

A second difficulty we face when attempting to apply minimum bounding limits is that such limits can rarely be established on physical grounds. This is especially true when using parameter tolerance limits. It is virtually impossible to imagine a situation where design engineers have somehow been able to precisely identify the minimum limits that bound values that are physically attainable. If we add to this the fact that tolerance limits are often influenced by marketing rather than engineering considerations, equating tolerance limits with minimum bounding limits becomes a very unfruitful and misleading practice.

100% Containment Criterion

By definition, the establishment of minimum bounding limits implies the establishment of 100% containment. It should be said however, that an uncertainty estimate may still be obtained for the uniform distribution if a containment probability less that 100% is applied. For instance, suppose the containment limits are given as $\pm L$ and the containment probability is stated as being equal to some value p between zero and one. Then, if the uniform probability criterion is met, the limits of the distribution are given by Eq. (9)

However, if the uniform probability criterion is not met the uniform distribution would still not be applicable, and we should turn to other distributions.

Uniform Probability Criterion

As discussed above, establishing minimum containment limits can be a challenging prospect. Harder still is finding real-world measurement error distributions that demonstrate a uniform probability of occurrence between two limits and zero probability of occurrence outside these limits. Except in very limited instances, such as are discussed in the next section, assuming a uniform probability is just not physically realistic. This is true even in some cases where the distribution would appear to be applicable.

For example, a conjecture has recently been advanced that the distribution of parameters immediately following test or calibration can be said to be uniform. While this seems reasonable at face value, it turns out not to be the case. Because of false accept risk (consumer's risk), such distributions range from approximately triangular to having a "humped" appearance with rolled-off shoulders.

As to whether we can treat parameter tolerance limits as bounds that contain values with uniform probability, we must imagine that, not only has the instrument manufacturer managed to miraculously ascertain minimum bounding limits, but has also juggled physics to such an extent as to make the parameter value's probability distribution uniform between these limits and zero outside them. This would be a truly amazing feat of engineering for most toleranced quantities — especially considering the marketing influence mentioned earlier.

Applicability of the Uniform Distribution

While the uniform distribution is not physically applicable to most error sources, there is a limited number of cases in which it meets the criteria described above.

Digital Resolution Uncertainty

We sometimes need to estimate the uncertainty due to the resolution of a digital readout. For instance, a three-digit readout might indicate 12.015 V. If the device employs the standard round-off practice, we know that the displayed number is derived from a sensed value that lies between 12.0145 V and 12.0155 V. We also can assert to a very high degree of validity that the value has an equal probability of lying anywhere between these two numbers. In this case, the use of the uniform distribution is appropriate, and the resolution uncertainty is

$$u_V = \frac{0.0005 \text{ V}}{\sqrt{3}} = 0.00029 \text{ V} .$$

RF Phase Angle

RF power incident on a load may be delivered to the load with a phase angle θ between $-\pi$ and π. In addition, unless there is a compelling reason to believe otherwise, the probability of occurrence between these limits is uniform. Accordingly, the use of the uniform distribution is appropriate. This yields a phase angle uncertainty estimate of

$$u_\theta = \frac{\pi}{\sqrt{3}} \cong 1.814 .$$

It is interesting to note that, given the above, if we assume that the amplitude of the signal is sinusoidal, the distribution for incident voltage is the U-shaped distribution.

Quantization Error

The potential drop (or lack of a potential drop) sensed across each element of an A/D Converter sensing network produces either a "1" or "0" to the converter. This response constitutes a "bit" in the binary code that represents the sampled value. For ladder-type networks, the position of the bit in the code is determined by the location of its originating network element.

Even if no errors were present in sampling and sensing the input signal, errors would still be introduced by the discrete nature of the encoding process. Suppose, for example, that the full scale signal level (dynamic range) of the A/D Converter is a volts. If n bits are used in the encoding process, then a voltage V can be resolved into 2^n discrete steps, each of size $a/2^n$. The error in the voltage V is thus

$$\varepsilon(V) = V - m\frac{a}{2^n} ,$$

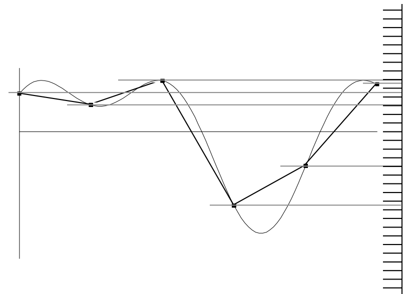
where m is some integer determined by the sensing function of the D/A Converter.

The containment limit associated with each step is one-half the value of the magnitude of the step. Consequently, the containment limit inherent in quantizing a voltage V is $(1/2)(a/2^n)$, or $a/2^{n+1}$. This is embodied in the expression

$$V_{quantized} = V_{sensed} \pm \frac{a}{2^{n+1}} .$$

The uncertainty due to quantization error is obtained from the containment limits and from the assumption that the sensed analog value has equal probability of occurrence between these limits:

$$u_V = \frac{a / 2^{n+1}}{\sqrt{3}} .$$



Signal Quantization. The sampled signal points are quantized in multiples of a discrete step size.

Important Considerations

Despite its "unphysical" character, applying the uniform distribution to obtain Type B uncertainty estimates is a practice that has been gaining ground over the past few years. There are two main reasons for this:

Ease of Use

Applying the uniform distribution makes it easy to obtain an uncertainty estimate. If the limits $\pm a$ of the distribution are known, the uncertainty estimate is computed using Eq. (8).

It should be said that the "ease of use" advantage should be tempered by the following warning:[6]

> "When a component of uncertainty is determined in this manner contributes significantly to the uncertainty of a measurement result, it is prudent to obtain additional data for its further evaluation."

At our present level of analytical development [SC04], the ease of use advantage is more difficult to justify than previously. Given the emergence of new methods and tools, the only "excuse" for not using a physically realistic distribution, applying a containment probability and estimating the degrees of freedom (if appropriate) is that the information for doing so is either (1) not at our fingertips or (2) it puts too much of a strain on technicians that are not trained in estimating uncertainty.[7]

GUM Authority

It has been asserted by some that the use of the uniform distribution is (uniformly?) recommended in the GUM. This is not true. In fact, most of the methodology of the GUM is based on the assumption that the underlying error distribution is normal.

Another source of confusion is that some of the examples in the GUM apply the uniform distribution in situations that appear to be incompatible with its use. It is reasonable to suppose that much of this is due to the fact that rigorous Type B estimation methods and tools were not available at the time the GUM was published, and the uniform distribution was an "easy out." As stated above, the lack of such methods and tools has since been rectified.

For clarification on this issue, the reader is referred to Section 4.3 of the GUM.

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[6] Quoted from Section 4.3.7 of the GUM.

[7] Since the goal of ensuring accurate measurements often involves the investment of considerable management and engineering time, equipment acquisition expenditures and proficiency training costs, it seems curious that, just when we get to the payoff, we become lazy and sloppy.

Developing Expanded Uncertainty Limits for Uniformly Distributed Errors

In recent years, it has become common practice to estimate expanded uncertainty limits for Type B and mixed estimates by multiplying the uncertainty estimate by a fixed "*k*-factor," usually equal to two. Assuming an underlying normal distribution, this produces limits that are roughly analogous to 95% confidence limits. The advisability of this practice is debatable, but that is the subject of a separate discussion. For the present, we consider what results from the practice when estimating an uncertainty for an error source where the underlying distribution is assumed to be uniform.

Since the uncertainty is estimated by dividing the distribution minimum bounding limit by the square root of three, multiplying this estimate by two yields expanded uncertainty limits that lie *outside* the distribution's minimum bounding limits. Curiously, these limits equate to approximately 115% containment probability, which is nonsense.

One way of reconciling the practice is to state that the underlying distribution is actually normal, or approximately normal, and the uniform distribution is used merely as an artifice to obtain an estimate of the distribution's standard deviation. This is a somewhat amazing statement. If the underlying distribution is normal, why not obtain the uncertainty estimate using that distribution in the first place? [8]

It can be shown that using the uniform distribution as a tool for estimating the uncertainty in a normally distributed quantity corresponds to assuming a normal distribution with a 91.67% containment probability. For organizations that maintain a high in-tolerance probability at the unit level, we often see or can surmise 98% or better in-tolerance probabilities at the parameter level. Consequently, for these cases, use of the uniform distribution produces uncertainty estimates that are at least 35% larger than what is appropriate.

As for those who find this acceptable on the basis of conservatism, consider the 72% end-of-period reliability target applied by one of the U.S. armed forces for general purpose items. For single-parameter items, if the true underlying distribution is normal, use of the uniform distribution produces uncertainty estimates that are only about 62% of what they should be. So much for conservatism.

THE LOGNORMAL DISTRIBUTION[9]

The lognormal distribution can often be used to estimate the uncertainty in equipment parameter bias in cases where the tolerance limits are asymmetric. It is also used in cases where a physical limit is present that lies close enough to the nominal or mode value to skew the parameter bias pdf in such a way that the normal distribution. is not applicable.

The pdf for a lognormally distributed variable *x* is given by

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[8] One recommendation that the reader may encounter is that, if all that is available for an error source or parameter deviation is a set of bounding limits, without any knowledge of the nature of the error distribution and with no information regarding a containment probability, then the uniform distribution should be assumed. There are two points that should be made concerning this recommendation.

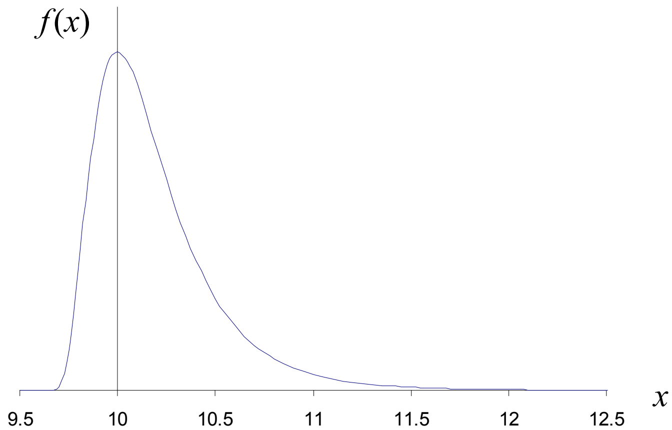
First, after a little reflection on the difficulty of obtaining minimum containment limits without knowledge of a containment probability, we can see that the recommendation not advisable. The prudent path to follow is to simply put some effort into obtaining a containment probability estimate and ascertaining a most likely underlying distribution. There is really no way around this. Moreover, the author has yet to observe an uncertainty analysis problem where this could not be done.

The second point is that, experienced technical personnel nearly always know *something* about what they are measuring and what they are measuring it with. Except for the cases described above, it is difficult to imagine a scenario where an experienced engineer or technician would know a set of bounding limits and nothing else.

[9] For a more complete discussion see the article "The Lognormal Distribution," at www.isgmax.com.

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma|x-q|}\exp\left\{-\left[\ln\left(\frac{x-q}{m-q}\right)\right]^2\Big/2\sigma^2\right\}, \tag{11}$$

where q is a physical limit for x and m is the population median. The variable σ is not the population standard deviation. It is referred to as the "shape parameter." The accompanying graphic shows a case where the population mode μ = 10, $q = 9.6207$, σ = 0.52046, and m = 10.8011. The computed standard deviation for this example is $u = 0.3176$.



The Lognormal Distribution. Useful for describing distributions for parameters constrained by a physical limit or possessing asymmetric tolerances. The case shown is a "right-handed" distribution in which the mode value is greater than the physical limit.

Distribution Statistics

In general, for a lognormally distributed variable, we have the following relations:

Mode	$\mu = q+(m-q)e^{-\sigma^2}$
Mean	$q+(m-q)e^{\sigma^2/2}$
Median	$m = q+(\mu-q)e^{\sigma^2}$
Variance	$(m-q)^2 e^{\sigma^2}(e^{\sigma^2}-1)$
Standard Deviation	$\|m-q\|e^{\sigma^2/2}\sqrt{e^{\sigma^2}-1}$

The quantities m, q and σ are obtained by numerical iteration, given containment limits and containment probabilities. To date, the only known analytical metrology applications that perform this process are UncertaintyAnalyzer [ISG04] and AccuracyRatio [ISG03a].

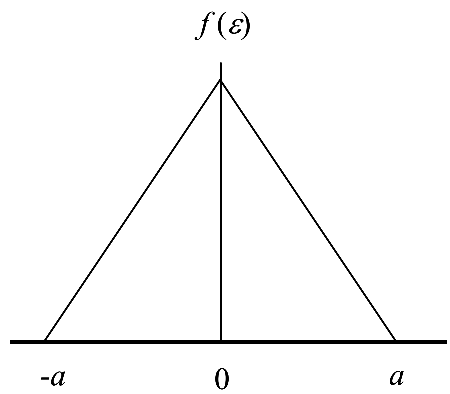
THE TRIANGULAR DISTRIBUTION

The triangular distribution has been proposed for use in cases where the containment probability is 100%, but there is a central tendency for values of the variable of interest [ISO97]. The triangular distribution is the simplest distribution possible with these characteristics.

The triangular distribution is also found in cases where two uniformly distributed errors with the same mean and bounding limits are combined linearly.

The pdf for the distribution is

$$f(\varepsilon)=\begin{cases}(\varepsilon+a)/a^2, & -a\le\varepsilon\le 0\\ (a-\varepsilon)/a^2, & 0\le\varepsilon\le a\\ 0, & \text{otherwise.}\end{cases}$$



The Triangular Distribution. A distribution for linear combinations of two uniformly distributed variables with equal mean values and bounding limits.

The standard deviation for the distribution is obtained from

$$u=\frac{a}{\sqrt{6}}\,. \tag{12}$$

Like the uniform distribution, using the triangular distribution requires the establishment of minimum containment limits $\pm a$. The same reservations apply in this regard to the triangular distribution as to the uniform distribution.

In cases where a containment probability $p < 1$ can be determined for limits $\pm L$, where $L < a$, the limits of the distribution are given by

$$a=\frac{L}{p}\left(1+\sqrt{1-p}\right),\quad L\le a\,. \tag{13}$$

Applicability of the Triangular Distribution

This distribution can be said to apply to linear combinations of pairs of uniformly distributed errors, as described above, and to post-test distributions under certain restricted conditions. It has also shown promise when applied to errors involved in interpolating between tabulated values. In these cases, the minimum error occurs at each tabulated value and the maximum error can often be assumed to occur at the mid point between these values. Moreover, assuming a linear error growth curve in these cases is methodologically tenable.

Apart from these cases, the triangular distribution has limited applicability to physical errors or deviations. While it does not suffer from the uniform probability criterion, as does the uniform distribution, it nevertheless displays abrupt transitions at the bounding limits and at the zero point, behaviors which are physically unrealistic for measurement errors. In addition, the linear slopes for probability of occurrence are somewhat fanciful for a pdf.

THE QUADRATIC DISTRIBUTION

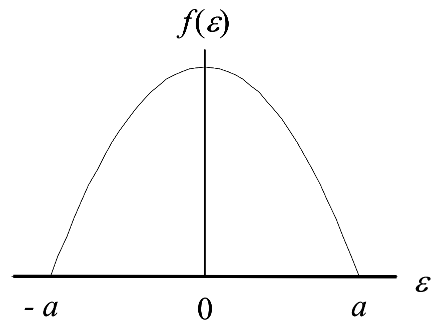
A distribution that eliminates the abrupt change at the zero point, does not exhibit unrealistic linear probability slopes and satisfies the need for a central tendency is the quadratic distribution. This distribution is defined by the pdf

$$f(\varepsilon) = \begin{cases} \frac{3}{4a}\left[1-(\varepsilon/a)^2\right], & -a \le \varepsilon \le a \\ 0, & \text{otherwise} \end{cases} \tag{14}$$

where ±*a* are minimum bounding limits. The standard deviation for this distribution is determined from

$$u = \frac{a}{\sqrt{5}}, \tag{15}$$

i.e., about 77% of the standard deviation estimate for the uniform distribution.



The Quadratic Distribution. Exhibits a central tendency without discontinuities and does not assume linear pdf behavior.

For a containment probability *p* and containment limits ±*L*, the minimum bounding limits ±*a* are obtained from

$$a = \frac{L}{2p}\left(1 + 2\cos\left[\frac{1}{3}\arccos(1-2p^2)\right]\right) \quad -1 < p < 1.$$

Applicability of the Quadratic Distribution

The quadratic distribution can be applied to error sources whose errors are known to have finite symmetric bounding limits and tend to not aggregate closely around the mid point of these limits. While it displays abrupt transitions at the bounding limits as do the uniform and triangular distributions, it is continuous at the zero point, and, therefore, is considered to be a more physically realistic distribution.

THE COSINE DISTRIBUTION

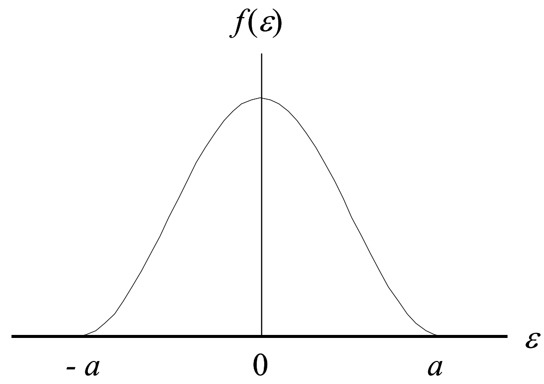
While the quadratic distribution eliminates discontinuities within the bounding limits, it rises abruptly *at* the limits. Although the quadratic distribution has wider applicability than either the triangular or uniform distribution, this feature nevertheless diminishes its physical validity. A distribution that overcomes this shortcoming, exhibits a central tendency and can be determined from minimum containment limits is the cosine distribution. The pdf for this distribution is given by

$$f(\varepsilon) = \begin{cases} \frac{1}{2a}\left[1+\cos\left(\frac{\pi\varepsilon}{a}\right)\right], & -a \le \varepsilon \le a. \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

The uncertainty is obtained from the expression

$$u = \frac{a}{\sqrt{3}}\sqrt{1-\frac{6}{\pi^2}}, \tag{17}$$

which translates to roughly 63% of the value obtained using the uniform distribution.



The Cosine Distribution. A 100% containment distribution with a central tendency and lacking discontinuities.

Solving for a when a containment probability p and containment limits $\pm L$ are given requires applying numerical iterative method to the expression

$$\frac{a}{\pi}\sin(\pi L/a) - ap + L = 0, \quad L \le a \;.$$

The solution algorithm has been implemented in the same software alluded to in the discussion on the quadratic distribution. It yields, for the ith iteration,

$$a_i = a_{i-1} - F / F' ,$$

where

$$F = \frac{a}{\pi}\sin(\pi L/a) - ap + L$$

and

$$F' = \frac{1}{\pi}\sin(\pi L/a) - \frac{L}{a}\cos(\pi L/a) - p \;.$$

Applicability of the Cosine Distribution

The cosine distribution can be applied to error sources whose errors are known to have finite symmetric bounding limits and display a tendency to aggregate around the mid point. Unlike the uniform, triangular or quadratic distributions, the cosine distribution is continuous both throughout its range and at the bounding limits. In appearance, it resembles the normal distribution and is considered to be physically realistic.

THE HALF-COSINE DISTRIBUTION

The half-cosine distribution is used in cases where the central tendency is not as pronounced as when normal or the cosine distribution would be appropriate. In this regard, it resembles the quadratic distribution without the discontinuities at the distribution limits. The pdf is

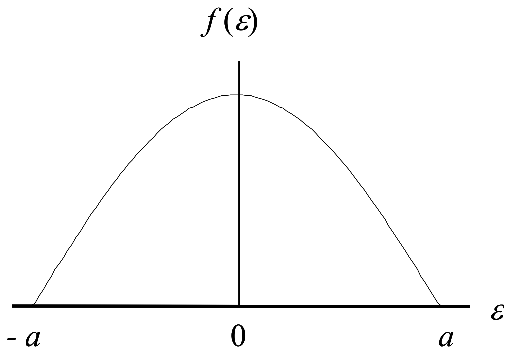
$$f(\varepsilon) = \begin{cases} \dfrac{\pi}{4a}\cos\left(\dfrac{\pi\varepsilon}{2a}\right) , & a \le \varepsilon \le a \;. \\ 0 , \quad \text{otherwise} \end{cases} \tag{18}$$

If the minimum limiting values $\pm a$ are known, the uncertainty is obtained from the expression

$$u = \sqrt{1 - 8/\pi^2}\; a \;. \tag{19}$$

If containment limits ±L and a containment probability p are known, the limiting values may be obtained from the relation

$$a = \frac{\pi L}{2\sin^{-1}(p)}, \quad L \le a\,. \tag{20}$$



The Half-Cosine Distribution. Possesses a central tendency but exhibits a higher probability of occurrence near the minimum limiting values than either the cosine or the normal distribution.

Applicability of the Half-Cosine Distribution

As stated above, the half-cosine distribution can be applied to error sources whose errors are known to have finite symmetric bounding limits but do not display a tendency to aggregate closely around the mid point. It displays the appearance of the quadratic distribution without the drawback of being discontinuous at the bounding limits.

Since the half-cosine distribution is continuous both throughout its range and at the bounding limits, it is more physically palatable than the uniform, triangular or quadratic distributions,.

THE U DISTRIBUTION

The following is a derivation of the U distribution, useful for specifying sinusoidal phase-varying subject parameters.

Preliminaries

The U distribution indicates the probability of sampling a particular sine wave amplitude from a signal whose phase is uniformly distributed. The relevant expressions are

$$\varepsilon = a\sin\varphi\,,$$

where the phase angle distribution is

$$h(\varphi) = \begin{cases} 1/2\pi, & -\pi \le \varphi \le \pi \\ 0, & \textit{otherwise.} \end{cases}$$

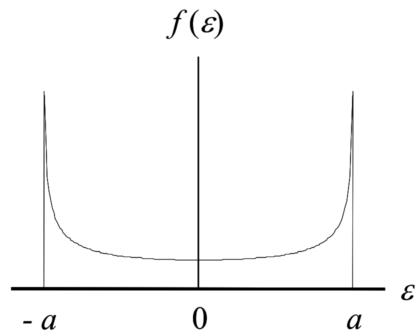
Change of Variable

The pdf for ε can be obtained by using the change of variable technique:

$$f(\varepsilon) = \left|\frac{d\varphi}{d\varepsilon}\right| h(\varepsilon(\varphi))\,,$$

which yields

$$f(\varepsilon) = \begin{cases} \dfrac{1}{\pi\sqrt{a^2-\varepsilon^2}}, & -a < x < a \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$



The U Distribution. The distribution is the pdf for sine waves of random phase incident on a plane.

The uncertainty in the incident signal amplitude is estimated according to

$$u = \frac{a}{\sqrt{2}} . \tag{22}$$

If containment limits $\pm L$ ($L < a$) and a containment probability $p < 1$ are known, the parameter a can be estimated according to

$$a = \frac{L}{\sin\left(\pi p / 2\right)}, \quad L \le a . \tag{23}$$

Applicability of the U Distribution

The U distribution applies to errors that vary sinusoidally with time, such as RF signals incident on a load or temperatures maintained by automatic environmental control systems. Since the distribution is derived from the properties of such phenomena, it is a physically realistic distribution.

STUDENT'S t DISTRIBUTION

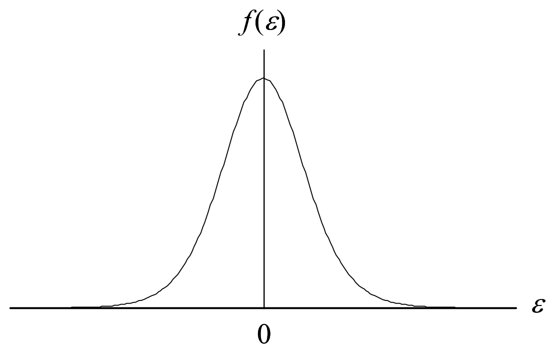
If an underlying error distribution is normal, and the degrees of freedom are available, confidence limits for measurement errors or parameter deviations may be obtained using the Student's t distribution. This distribution is available in statistics textbooks and popular spreadsheet applications. Its pdf is

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)}(1 + x^2/\nu)^{-(\nu+1)/2} , \tag{24}$$

where ν is the degrees of freedom and $\Gamma(\cdot)$ is the gamma function.

The degrees of freedom quantifies the amount of knowledge used in estimating uncertainty. For a Type B estimate, this knowledge is incomplete if the limits $\pm a$ are approximate and the containment probability p is estimated from recollected experience. Since the knowledge is incomplete, the degrees of freedom associated with a Type B estimate is not infinite.

If the underlying error distribution is normal, the degrees of freedom can be estimated using Eq. (7). If the degrees of freedom can be determined for a Type B estimate, the uncertainty can be rigorously combined with Type A estimates.



Student's t Distribution. Shown is the pdf for zero mean and 10 degrees of freedom.

Once the degrees of freedom has been obtained, the degrees of freedom for the combined uncertainty can be determined using the Welch-Satterthwaite relation [ISO97]. If the underlying distribution for the combined estimate is normal, the *t*-distribution can be used to develop confidence limits and perform statistical tests [HC00].

THE TRAPEZOIDAL DISTRIBUTION

If two errors ε_x and ε_y are uniformly distributed with bounding values b and $a > b$, then their sum

$$\varepsilon = \varepsilon_x + \varepsilon_y$$

follows a trapezoidal distribution with discontinuities at $\pm (a + b)$ and $\pm (a - b)$. The pdf is given by

$$f(\varepsilon) = \begin{cases} \frac{1}{4ab}(a+b+\varepsilon)\,, & -(a+b) \le \varepsilon \le -(a-b) \\ \frac{1}{2b}\,, & -(a-b) \le \varepsilon \le a-b \\ \frac{1}{4ab}(a+b-\varepsilon)\,, & a-b \le \varepsilon \le a+b \\ 0\,. & \text{otherwise} \end{cases} \tag{25}$$

Integrating ε^2 over the bounding limits of this distribution yields the variance

$$\text{var}(\varepsilon) = \frac{a^2 + b^2}{3}\,,$$

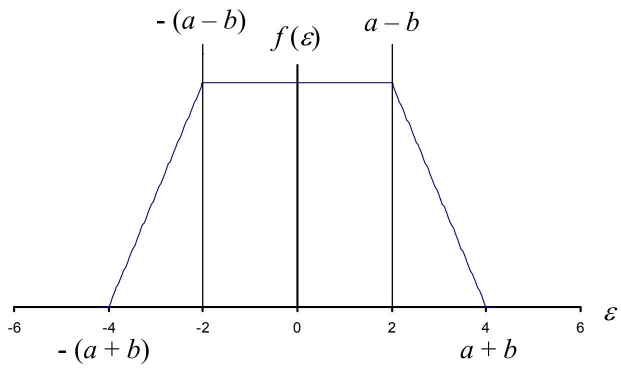
and the standard uncertainty is then

$$u_\varepsilon = \sqrt{\frac{a^2 + b^2}{3}} \tag{26}.$$

Applicability of the Trapezoidal Distribution

As will be shown later in the discussion on convolved errors, the trapezoidal applies to the sum of two uniformly distributed errors. Apart from this fact, it is difficult to imagine an instance where it would be applicable on it own merit. It has been recommended in cases where 100% containment limits are known, the probability density is believed to be less near the limits than at their midpoint and there exists a region inside the limits where the pdf is approximately uniform.

Given these considerations, we might be more inclined to use the quadratic, cosine, half-cosine or utility (see below) distribution on the grounds that each exhibits a more physically realistic character. Another point in its disfavor is that the construction of the trapezoidal distribution requires not only the specification of 100% containment limits but uniform density limits as well.

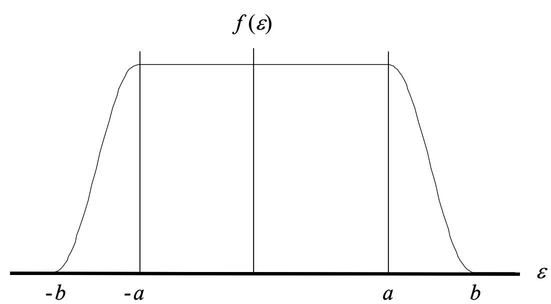


Trapezoidal Distribution. Shown is a distribution for the sum of two uniformly distributed variables with bound values $a = 3$ and $b = 1$. The 100% containment limits are $\pm (a + b)$ and the uniform density limits are $\pm (a - b)$.

In the author's experience, obtaining the limits of the trapezoidal distribution from technical expertise has been met with dubious success.

THE UTILITY DISTRIBUTION

One of the key variables in evaluating the return on investment of alternative technical decisions or policies is a quantity called *utility*. A utility function that has been applied to evaluating ROI for test and calibration support hierarchies [HC89] is one whose behavior is somewhat similar to that of the trapezoidal distribution. Specifically, this function has a region of approximately uniform utility, bounded by limits that mark points where the utility begins to decrease from its maximum value, eventually reaching limits that correspond to zero utility.



The Utility Distribution. The probability of occurrence is approximately uniform between the values $\pm a$, tapering off to zero at the limits $\pm b$.

The pdf for the utility distribution is

$$f(\varepsilon) = \begin{cases} \dfrac{1}{a+b}, & |\varepsilon| \le a \\ \dfrac{1}{a+b}\cos^2\left[\dfrac{\pi(|\varepsilon|-a)}{2(b-b)}\right], & a \le |\varepsilon| \le b \\ 0\,, & |\varepsilon| \ge b\,, \end{cases} \tag{27}$$

where a and b are as shown in the graphic. An example of the pdf for this distribution is shown above.

Integrating ε^2 over the bounding limits of this distribution yields the variance

$$\text{var}(\varepsilon) = \frac{a^3 + b^3}{3(a+b)} - \frac{2}{\pi^2}(b-a)^2 ,$$

and the standard uncertainty is then

$$u = \sqrt{\frac{a^3 + b^3}{3(a+b)} - \frac{2}{\pi^2}(b-a)^2} .$$

Applicability of the Utility Distribution

The character of this function makes it arguably more useful than the trapezoidal distribution in that it is free of discontinuities and requires the same information for its specification. While obtaining the necessary information for constructing a utility function is fairly straightforward, obtaining the limits $\pm a$ for the utility distribution is hampered by the same practical difficulties encountered in obtaining the limits for the uniform density portion of the trapezoidal distribution.

RECOMMENDATIONS FOR SELECTING AN ERROR SOURCE DISTRIBUTION

The following are offered as guidelines for selecting an appropriate error distribution:

1. Unless information to the contrary is available, the normal distribution should be applied as the default distribution. For Type B estimates, the degrees of freedom should be estimated using Eq. (7) and the Student's t distribution should be employed to develop confidence limits and to perform statistical tests.

2. If it is suspected that the distribution of the value of interest is skewed, apply the lognormal distribution.

   In using the normal or lognormal distribution, some effort must be made to estimate a containment probability. If a set of containment limits is available, but 100% containment has been observed, then the following is recommended:

3. If the value of interest has been subjected to random usage or handling stress, and is assumed to possess a central tendency, apply the cosine distribution. If it is suspected that values are more evenly distributed, apply either the quadratic or half-cosine distribution, as appropriate. The triangular distribution may be applicable to estimating uncertainty due to interpolation errors, and, under certain circumstances, when dealing with parameters following testing or calibration.
4. If the value of interest is the amplitude of a sine wave incident on a plane with random phase, apply the U distribution.
5. If the value of interest is the resolution uncertainty of a digital readout, apply the uniform distribution. This distribution is also applicable to estimating the uncertainty due to quantization error and the uncertainty in RF phase angle.

GENERAL PROCEDURES FOR OBTAINING AN UNCERTAINTY ESTIMATE FOR AN ERROR SOURCE

Type A Estimates

In making a Type A estimate and using it to construct confidence limits, we apply the following procedure taken from the GUM and elsewhere:

1. Take a random sample of size n representative of the population of interest. The larger the sample size, the better. In many cases, a sample size less than six is not sufficient.
2. Compute a sample standard deviation, u using the relation

   $$u_x = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2} ,$$

   where the x_i, $i = 1, 2, \cdots, n$ comprise a sample of n measured values.
3. Assume an underlying distribution, e.g., normal.

4. Develop a coverage factor based on the degrees of freedom (n – 1) associated with the sample standard deviation and a desired level of confidence. If the underlying distribution is assumed to be normal, use either t-tables or Student's t spreadsheet functions. In Microsoft Excel, for example, a two-sided coverage factor can be determined using the TINV function: $t = \text{TINV}((1-p),\nu)$, where p is the confidence level and ν is the degrees of freedom
5. Multiply the sample standard deviation by the coverage factor to obtain $L = tu$ and use ±L as $p \times 100\%$ confidence limits.

Type B Estimates

In making a Type B estimate, we reverse the process. The procedure is

1. Develop a set of error containment limits ±L.
2. Estimate a containment probability p.
3. Estimate the degrees of freedom as described in [HC05].
4. Assume an underlying distribution, e.g., normal.[10]
5. Compute a coverage factor, t, based on the containment probability and degrees of freedom.
6. Compute the standard uncertainty for the quantity of interest (e.g., parameter bias) by dividing the confidence limit by the coverage factor: $u = L/t$.

COMBINING UNCERTAINTIES - THE STANDARD METHOD

As discussed earlier, the standard uncertainty in a measurement can be determined using Eqs. (2) and (3). These expressions are valid, regardless of the error distribution or measurement classification. The standard method takes advantage of this and invokes the central limit theorem mentioned earlier. This allows us to treat combinations of errors as normally distributed variables, thereby making available the array of tools developed for this distribution in computing confidence limits, testing hypotheses, developing tolerances, and so on.

DIRECT MEASUREMENTS

A direct measurement is one in which the value of the measurand is sought by measuring a single quantity. An example of a direct measurement is the measurement of the voltage of a DC battery using a voltmeter.

For direct measurements, the relevant error model for a measurement subject to k different error sources is

$$\varepsilon = \sum_{i=1}^{k} \varepsilon_i \ . \tag{28}$$

By Eq. (2) the variance of an error distribution is the square of the standard uncertainty of the error. Using Eq. (2), It can be shown that the variance of ε is given by

$$\text{var}(\varepsilon) = \sum_{i=1}^{k} \text{var}(\varepsilon_i) + 2\sum_{i=1}^{k-1}\sum_{j>i}^{k} \text{cov}(\varepsilon_i, \varepsilon_j) \ , \tag{29}$$

where var(ε_i) is as defined earlier, and where the cov(·) term represents the *covariance* between error sources. Ordinarily, we replace the covariance term with a *correlation coefficient* defined by

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[10] The Type B estimation procedure has been refined so that standard deviations can be estimated for non-normal populations and in cases where the confidence limits are asymmetric or even single-sided [3, 5].

$$\rho_{ij} = \frac{\operatorname{cov}(\varepsilon_i, \varepsilon_j)}{u_i u_j}, \quad (30)$$

where

$$u_i = \sqrt{\operatorname{var}(\varepsilon_i)}, \quad i = 1, 2, \cdots, k \quad (31)$$

Substituting Eq. (30) in Eq. (29) and using Eq. (3) gives

$$\operatorname{var}(\varepsilon) = \sum_{i=1}^{k} u_i^2 + 2\sum_{i=1}^{k-1}\sum_{j>i}^{k} \rho_{ij} u_i u_j .$$

For direct measurements, the errors are nearly always statistically independent, and the correlation coefficients are zero. The expression

$$u_\varepsilon = \sqrt{\sum_{i=1}^{k} u_i^2} \text{ , s-independent errors} \quad (32)$$

applies in these cases. Invoking the central limit theorem, we treat ε as a normally distributed quantity, and use the Student's t distribution and other statistical tools in applying the uncertainty estimate u_ε.

Degrees of Freedom

The amount of information used to estimate the uncertainty in a given error is called the *degrees of freedom*. The degrees of freedom is required, among other things, to employ an uncertainty estimate in computing confidence limits commensurate with some desired confidence level.

The degrees of freedom for a Type A estimate is usually taken to be n - 1, where n is the number of independent measurements employed in computing the Type A uncertainty estimate. The degrees of freedom for a Type B estimate can be determined using the methodology of [HC05].

Whether estimates are Type A or Type B, the degrees of freedom for a combined uncertainty estimate for an error composed of a linear combination of errors is given by the Welch-Satterthwaite relation. For uncorrelated errors, this relation is given by

$$\nu = \frac{u^4}{\sum_{i=1}^{n} \frac{u_i^4}{\nu_i}},$$

where u is the combined uncertainty and u_i and ν_i are the uncertainty and degrees of freedom, respectively, for the ith uncertainty estimate, $i = 1, 2, \cdots, k$.

If errors are correlated, a modified form of the Welch-Satterthwaite relation has been recently developed [HC05]:

$$\nu = \frac{u^4}{\sum_{i=1}^{n} \frac{u_i^4}{\nu_i} + 2\sum_{i=1}^{n-1}\sum_{j>i}^{n} \rho_{ij}^2 \sigma^2(u_i)\sigma^2(u_j)},$$

where ρ_{ij} if the correlation coefficient for the ith and jth errors and $\sigma^2(u)$ is given in [HC05].

MULTIVARIATE MEASUREMENTS

With multivariate measurements, two or more quantities are measured to obtain the value of the measurand. For example, the measurements of the area of a rectangular plate would involve measurements of the plate's length and width.

In multivariate measurements, we work with a *system equation* that describes the measurement result in terms of the individual component measurements. To illustrate, let y represent the value of the measurand and let $x_1, x_2, \cdots, x_m$ represent the quantities measured to obtain a value for y. In the parlance of uncertainty analysis, these quantities are referred to as *components* of the measurement, and their associated measurement errors are called *error components*.

The system equation would be written

$$y = f(x_1, x_2, \cdots, x_m) ,$$

or, alternatively,

$$y = f(\mathbf{x}) ,$$

where

$$\mathbf{x} = (x_1, x_2, \cdots, x_m)^T .$$

Before we can compute a variance for the error in the measurement of y, we need to develop an error model. The usual approach is to expand y in terms of its components using a first-order MacLaurin series:

$$\varepsilon = \sum_{i=1}^{m} \left(\frac{\partial y}{\partial x_i} \right) \varepsilon_i . \tag{33}$$

Next, we apply Eq. (3) to obtain

$$\mathrm{var}(\varepsilon) = \sum_{i=1}^{m} \left(\frac{\partial y}{\partial x_i} \right)^2 u_i^2 + 2 \sum_{i=1}^{m-1} \sum_{J>i}^{m} \rho_{ij} \left(\frac{\partial y}{\partial x_i} \right) \left(\frac{\partial y}{\partial x_j} \right) u_i u_j . \tag{34}$$

In this expression, the correlation coefficient represents *cross-correlations* between error components. These coefficients are expressed in terms of the applicable error sources according to

$$\rho_{ij} = \frac{1}{u_i u_j} \sum_{k=1}^{n_k} \sum_{l=1}^{n_l} \rho_{ijkl} u_{ik} u_{jl} ,$$

where the terms u_{ik} represent the uncertainty due to the *kth* error source in the measurement of the *ith* error component and ρ_{ijkl} is the correlation coefficient for the kth error source of the ith error component and the lth error source of the jth error component, i.e.,

$$\rho_{ijkl} = \frac{\mathrm{cov}(\varepsilon_{ik}, \varepsilon_{jl})}{u_{ik} u_{jl}} .$$

The uncertainty u_i in the *ith* component error is obtained from

$$u_i^2 = \sum_{j=1}^{n_i} u_{ij}^2 + 2 \sum_{k=1}^{n_i - 1} \sum_{l>k}^{n_i} \rho_{ikl} u_{ik} u_{il} . \tag{35}$$

The coefficient ρ_{ikl} is the correlation coefficient for the kth and lth error sources of the ith error component

$$\varepsilon_i = \sum_{j=1}^{n_i} \varepsilon_{ij} ,$$

that is,

$$\rho_{ikl} = \frac{\mathrm{cov}(\varepsilon_{ik}, \varepsilon_{il})}{u_{ik} u_{il}} .$$

CONVOLVING S-INDEPENDENT ERROR SOURCES

If two errors are statistically independent, the distribution of their sum can be readily obtained by convolution. The procedure is as follows: Let ε_x and ε_y be two s-independent continuously distributed measurement errors with pdfs $f(\varepsilon_x)$ and $g(\varepsilon_y)$, respectively. Then the distribution of

$$\varepsilon = \varepsilon_x + \varepsilon_y \tag{36}$$

can be obtained from the relation

$$h(\varepsilon) = \int_{-\infty}^{\infty} f(\varepsilon_x) g(\varepsilon - \varepsilon_x) d\varepsilon_x \ . \tag{37}$$

Several examples of distributions obtained in this way are given in the next two sections

DIRECT MEASUREMENTS

A direct measurement is one in which the value of the measurand is sought by measuring a single quantity. An example of a direct measurement is the measurement of the voltage of a DC battery using a voltmeter.

For direct measurements, the relevant error model for a measurement subject to k different error sources is

$$\varepsilon = \sum_{i=1}^{k} \varepsilon_i \ . \tag{38}$$

In using this expression, we might imagine that ε_1 refers to equipment bias, ε_2 represents repeatability error, ε_3 is operator bias, etc. For practically all direct measurements, these error are s-independent and convolution can be used to obtain the distribution for ε. Expressions are provided in what follows for certain simple convolutions. For general cases, it is often preferable to integrate Eq. (37) numerically, as discussed later.

Convolved Uniform Distributions

Two-Variable Case

Consider two s-independent uniformly distributed errors ε_1 and ε_2 with pdfs

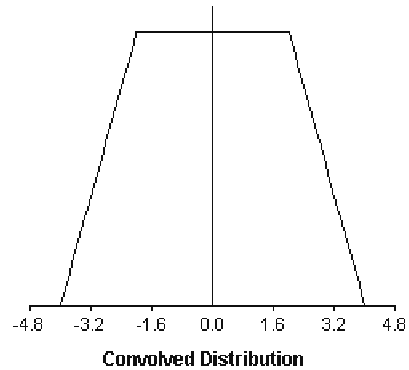
$$f(\varepsilon_1) = \begin{cases} \dfrac{1}{2a}, & -a \le \varepsilon_1 \le a \\ 0, & \text{otherwise}, \end{cases}$$

and (39)

$$g(\varepsilon_2) = \begin{cases} \dfrac{1}{2b}, & -b \le \varepsilon_2 \le b \\ 0, & \text{otherwise}. \end{cases}$$

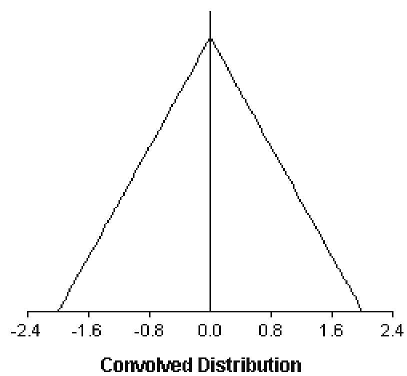
Using Eq. (39) in Eq. (37) gives for $b \ge a$

$$h(\varepsilon) = \begin{cases} \dfrac{1}{4ab}(a+b+\varepsilon)\,, & -(a+b) \le \varepsilon \le -(b-a) \\ \dfrac{1}{2b}\,, & -(b-a) \le \varepsilon \le b-a \\ \dfrac{1}{4ab}(a+b-\varepsilon)\,, & b-a \le \varepsilon \le b+a \\ 0\,. & \text{otherwise} \end{cases} \tag{40}$$



The Trapezoidal Distribution. Shown is the distribution for the sum of two uniformly distributed errors with equal means and where $b > a$. A freeware application was used to develop the graphic [ISG04b].

An interesting example in which two uniformly distributed errors are combined is when $a = b$. In this case, the distribution becomes the familiar triangular distribution shown below.



The Triangular Distribution. Shown is the distribution for the sum of two uniformly distributed errors with equal means and equal limits.

Three-Variable Cases

Consider three s-independent uniformly distributed errors ε_1, ε_2 and ε_3 with pdfs

$$w(\varepsilon_1) = \begin{cases} \dfrac{1}{2a}, & -a \le \varepsilon_1 \le a \\ 0, & \text{otherwise}, \end{cases}$$

$$g(\varepsilon_2) = \begin{cases} \dfrac{1}{2b}, & -b \le \varepsilon_2 \le b \\ 0, & \text{otherwise}, \end{cases}$$

and (41)

$$h(\varepsilon_3) = \begin{cases} \dfrac{1}{2c}, & -c \le \varepsilon_3 \le c \\ 0, & \text{otherwise}. \end{cases}$$

A number of combinations of relative sizes of a, b and c are possible. One case, where $c > b > a$ and $c > a + b$, will be explicitly constructed. Other cases can be built in like manner. All cases will be shown in the following figures.

Case 1: $c > b > a$ and $c > a + b$

The pdf for this case is broken up into six pieces:

$$f(\varepsilon) = \frac{1}{8abc}(c+b+a+\varepsilon)^2, \quad -(a+b+c) \le \varepsilon \le -(b-c)-a$$

$$f(\varepsilon) = \frac{1}{4ab}(a+b+\varepsilon), \quad -(b-c)-a \le \varepsilon \le (b-c)-a$$

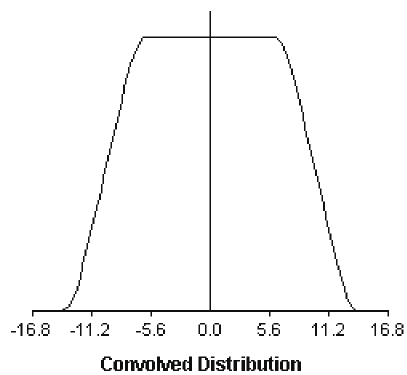
$$f(\varepsilon) = \frac{1}{8abc}\left[\frac{1}{2}(a+b+c+\varepsilon)^2 - (a+\varepsilon)^2 - (b-c)^2\right], \quad (b-c)-a \le (b+c)-a$$

$$f(\varepsilon) = \frac{1}{2a}, \quad (b+c)-a \le \varepsilon \le a-(b+c)$$

$$f(\varepsilon) = \frac{1}{8abc}\left[4bc - \frac{1}{2}(b+c-a+\varepsilon)^2\right], \quad a-(b+c) \le \varepsilon \le a-(b-c)$$

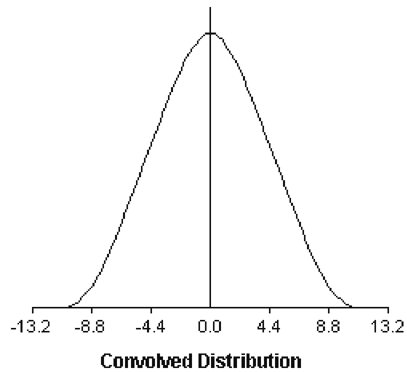
$$f(\varepsilon) = \frac{1}{8abc}(a+b+c-\varepsilon)^2, \quad (b-c)+a \le \varepsilon \le a+b+c\,.$$

The pdf for this combination of bounding limits is shown below.



An Approximate Utility Distribution. Shown is the distribution for the sum of three uniformly distributed errors with equal means and where $a > b > c$. In the case depicted, $c > a + b$. Notice the similarity in form of the convolved distribution and the utility distribution discussed earlier.

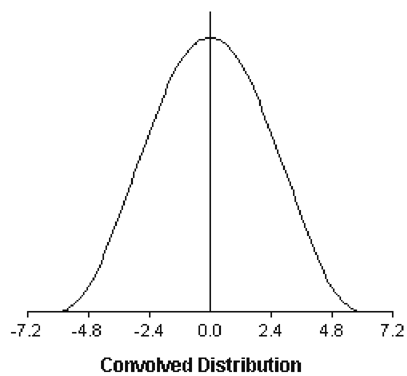
Case 2: *c* > *b* > *a* and *c* < *a* + *b*



An "Unbalanced" Uniform Convolution. Shown is the distribution for the sum of three uniformly distributed errors with equal means and where $a > b > c$. In the case depicted, $c < a + b$. In this case, the convolved distribution appears to favor both the normal and triangular distributions.

Case 3: *c* > *b* > *a* and *a* = *b* + *c*

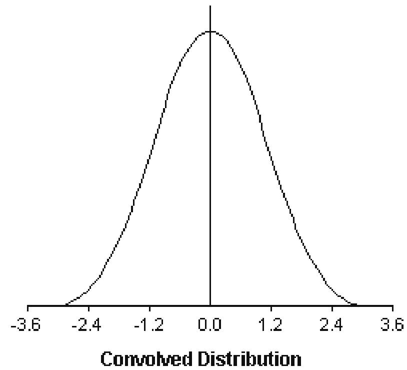
In this case, the convolved distribution takes on a more normal appearance, as shown below.



A "Symmetric" Uniform Convolution. Shown is the distribution for the sum of three uniformly distributed errors with equal means and where $a > b > c$. In the case depicted, $c = a + b$. In this case, the convolved distribution begins to take on the character of the normal distribution.

Case 4: *a* = *b* = *c*

In this case, the convolved distribution takes on a normal appearance, squeezed in the middle, as shown below.



A "Balanced" Uniform Convolution. Shown is the distribution for the sum of three uniformly distributed errors with equal means and where $a = b = c$. In this case, the convolved distribution looks like a normal distribution cinched at the waist

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Convolved Normal Distributions

Two-Variable Case

Consider two s-independent normally distributed errors ε_1 and ε_2 with pdfs

$$f(\varepsilon_1) = \frac{1}{\sqrt{2\pi u_1}} e^{-(\varepsilon-\mu_1)^2/2u_1^2},$$

and (42)

$$g(\varepsilon_2) = \frac{1}{\sqrt{2\pi u_2}} e^{-(\varepsilon-\mu_2)^2/2u_2^2}$$

Using Eq. (42) in Eq. (37) yields for $\varepsilon = \varepsilon_1 + \varepsilon_2$

$$h(\varepsilon) = \frac{1}{\sqrt{2\pi u}} e^{-(\varepsilon-\mu)^2/2u^2}, \tag{43}$$

where

$$u = \sqrt{u_1^2 + u_2^2},$$

and

$$\mu = \mu_1 + \mu_2 .$$

From Eq. (43), we see that the convolved distribution is itself a normal distribution. By induction, we note that the distribution of the sum of any number of normally distributed errors is also a normal distribution.

Convolved Uniform and Normal Distributions

For this combination, we have

$$f(\varepsilon_1) = \begin{cases} \dfrac{1}{2a}, & \mu_1 - a \le \varepsilon_1 \le \mu_1 + a \\ 0, & \text{otherwise.} \end{cases}$$

and

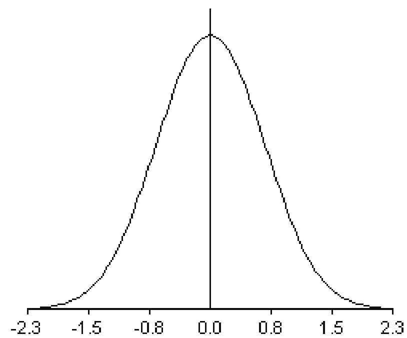
$$g(\varepsilon_2) = \frac{1}{2\pi u_2} e^{-(\varepsilon_2-\mu_2)^2/2u_2^2} .$$

Convolving these pdfs yields

$$h(\varepsilon) = \frac{1}{2a}\left\{\Phi\left(\frac{\varepsilon - \mu_2 - (\mu_1 - a)}{u_2}\right) - \Phi\left(\frac{\varepsilon - \mu_2 - (\mu_1 + a)}{u_2}\right)\right\},$$

where Φ is the normal distribution function defined by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt .$$



Convolution of Uniformly and Normally Distributed Errors. Shown is the distribution for the sum of a uniformly distributed error and a normally distributed error with mean values equal to zero. In the case depicted, u_1 and u_2 are equal.

Convolved Uniform and Cosine Distributions

For this combination, we have

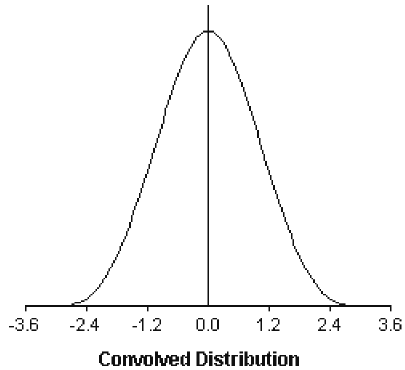
$$f(\varepsilon_1) = \begin{cases} \frac{1}{2a}, & \mu_1 - a \le \varepsilon_1 \le \mu_1 + a \\ 0, & \text{otherwise.} \end{cases}$$

and

$$g(\varepsilon_2) = \begin{cases} \frac{1}{2b}\left[1 + \cos\frac{\pi(\varepsilon - \mu_2)}{b}\right], & \mu_2 - b \le \varepsilon \le \mu_2 + b \\ 0, & \text{otherwise.} \end{cases}.$$

For simplicity, let $\mu_1 = \mu_2 = 0$. Then the convolution yields

$$h(\varepsilon) = \frac{1}{2b} + \frac{1}{4\pi a}\left\{\sin\left[\frac{\pi(\varepsilon + a)}{b}\right] - \sin\left[\frac{\pi(\varepsilon - a)}{b}\right]\right\}.$$



Convolution of Uniform and Cosine Distributions. Shown is the distribution for the sum of a uniformly distributed error and a an error that follows the cosine distribution. In the case depicted, mean values for both errors are zero and $b = 2a$.

MONTE CARLO METHODS

S-INDEPENDENT ERROR SOURCES

The errors associated with many of the measurements we make are statistically independent of one another. In what follows, we examine both direct measurement cases, in which a measuring device directly measures the quantity of interest, and multivariate measurement cases, in which several different quantities are measured to obtain the value of the quantity of interest.

Direct Measurements

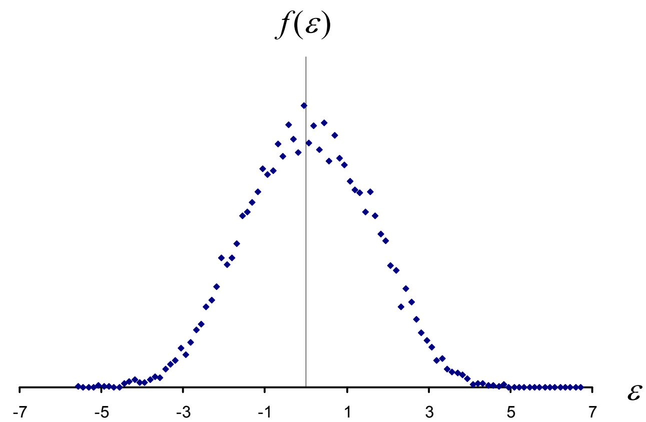
Let ε_1 and ε_2 represent s-independent errors involved in making a direct measurement. The error model becomes

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \tag{43}$$

To simulate a distribution of errors ε, we simulate an n-component vector of errors $\boldsymbol{\varepsilon}_1$ and an n-component vector of errors $\boldsymbol{\varepsilon}_2$ and add the two to get the vector of simulated errors $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2 \ . \tag{44}$$

A simulated distribution for ε is shown below.



Monte Carlo Simulation for the Distribution of the Sum of Two Uncorrelated Measurement Errors. Shown is the distribution for a measurement error composed of a uniformly distributed error with bounding limit a = 2.0 and a normally distributed error with a standard deviation of 1.0. The population standard deviation (standard uncertainty) is 1.5275. The standard deviation computed from the simulated distribution is 1.5238.

Multivariate Measurements

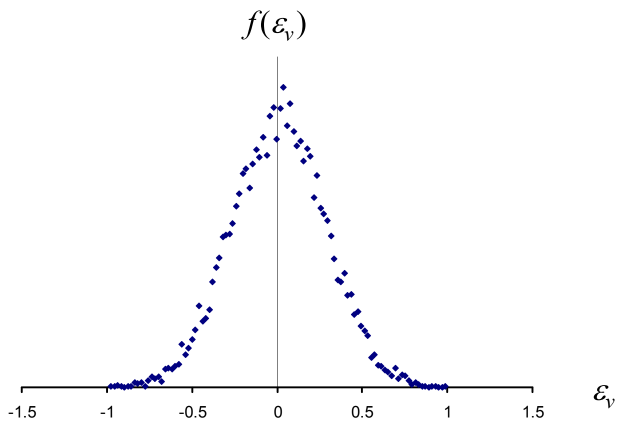
Consider the measurement of velocity obtained by measuring distance d and time t:

$$v = \frac{d}{t} \,. \tag{45}$$

Suppose that d is measured with a tape measure and that t is measured with a stopwatch. For discussion purposes, imagine that the error in the distance measurement is due almost entirely to bias in the tape measure and that the error in the time measurement is due almost entirely to bias in the stopwatch. Denote these errors ε_d and ε_t, respectively. Then the error model can be written

$$\varepsilon_v = \frac{d + \varepsilon_d}{t + \varepsilon_t} - v \,. \tag{46}$$

For this case, we simulate pairs of errors ε_d and ε_t, and place each pair in Eq. (46). An example of this simulation is shown below.



Monte Carlo Simulation for a Multivariate Measurement Involving Two Uncorrelated Measurement Errors. Shown is the distribution for the error ε_v in Eq. (46). In the simulation, a distance of 100 *m* was measured using a tape measure and an elapsed time of 10 *s* was measured using a stopwatch. The error in the distance measurement was assumed to be a normally distributed bias in the tape measure and the error in the time measurement was assumed to consist of a (uniformly distributed) digital resolution error. The standard deviation of the distance measurement was set at 2.5, while resolution limits of ±0.2 were used for the time measurement. The population standard deviation computed using the variance addition rule and small error theory is 0.2754 *m/s*. The standard deviation computed from the simulated distribution is 0.2751 *m/s*.

CORRELATED ERROR SOURCES

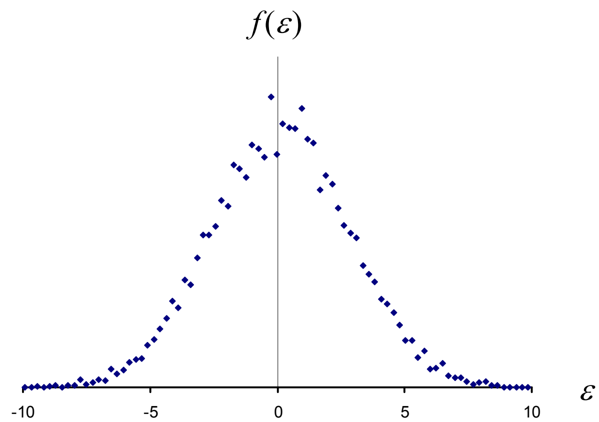
Direct Measurements

Let ε_1 and ε_2 represent errors involved in making a direct univariate measurement. Then, the error model is again given in Eq. (43)

If ε_1 and ε_2 are correlated with a correlation coefficient ρ, then, for each simulated value of ε_1, we simulate a value $\rho\varepsilon_2$. Likewise, for each simulated value of ε_2, we simulate a value $\rho\varepsilon_1$. Since we are double-counting, we divide by two. The prescription is just

$$\begin{aligned} \varepsilon &= \frac{1}{2}\left(\varepsilon_1 + \rho\varepsilon_2 + \varepsilon_2 + \rho\varepsilon_1\right) \\ &= \frac{1}{2}(1+\rho)(\varepsilon_1 + \varepsilon_2). \end{aligned} \tag{47}$$

$$\begin{aligned} \varepsilon &= \frac{1}{2}\left(\varepsilon_1 + \rho\varepsilon_2 + \varepsilon_2 + \rho\varepsilon_1\right) \\ &= \frac{1}{2}(1+\rho/3)(\varepsilon_1 + \varepsilon_2). \end{aligned}$$



Both normal. u1 = 1, u2 = 2, rho = 0.60
Simulated Sigma = 2.709560853
Population Sigma = 2.720294102

Multivariate Measurements

Consider the measurement of the area of a rectangular plate obtained by measuring length *a* and width *b*:

$$A = ab\ . \tag{48}$$

Suppose that *a* and *b* are measured with the same steel ruler. Then the length and width measurements are strongly correlated. Again, for discussion purposes, imagine that the errors in the measurements are due almost entirely to bias in the steel ruler. Denote these errors ε_a and ε_b, respectively. Then the error model can be written

$$\begin{aligned}\varepsilon_A &= (a+\varepsilon_a)(b+\varepsilon_b) - ab \\ &= a\varepsilon_b + b\varepsilon_a + \varepsilon_a\varepsilon_b.\end{aligned} \tag{48}$$

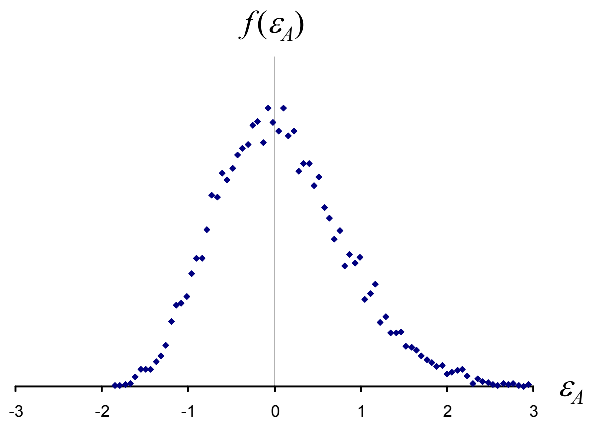
Let ρ be the correlation coefficient between the errors in measurement of *a* and *b*. Then, for each simulated value of ε_1, we simulate a value $\rho\varepsilon_2$. Likewise, for each simulated value of ε_2, we simulate a value $\rho\varepsilon_1$. Since we are double-counting, we divide the total by two and the error cross product by four. The prescription then becomes

$$\varepsilon_A = \frac{1}{2}\left[a(\varepsilon_b + \rho\varepsilon_a) + b(\varepsilon_a + \rho\varepsilon_b)\right] + \frac{1}{4}\left[\,(\varepsilon_a + \rho\varepsilon_b)(\varepsilon_b + \rho\varepsilon_a)\right]\ . \tag{49}$$

Suppose that the length and width measurements are made at the same time. Then the biases in both measurements are equal, $\rho = 1$, and Eq. (49) becomes

$$\varepsilon_A = \left(a+b\right)\varepsilon + \varepsilon^2\ . \tag{50}$$

A simulated distribution for ε_A is shown below.



Monte Carlo Simulation for the Distribution of Correlated Measurement Biases. Shown is the distribution for the error in the measurement of a rectangular plate of length 1.0 and width 2.0. In constructing the distribution, 10,000 random normal deviates were simulated from a population with mean zero and standard deviation 0.25. The population standard deviation is 0.75. The standard deviation computed from the simulated distribution is 0.7544.

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