Estimating Type B Degrees of Freedom Equation

Technical Note
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The key to estimating the degrees of freedom for a Type B uncertainty estimate lies in considering the distribution for a sample standard deviation for a sample with sample size *n*. We know that the degrees of freedom for the standard deviation estimate is $\nu = n - 1$.

Let s_ν represent the standard deviation, taken on a sample of size $n = \nu + 1$ of a $N(0, u^2)$ variable *x*. We know that the quantity $\nu s_\nu^2 / u^2$ is χ^2-distributed with ν degrees of freedom.

The χ^2-distribution has the pdf

$$f(x) = \frac{x^{(\nu-1)/2} e^{-x/2}}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)} \; .$$

Accordingly, we set $x = \nu s_\nu^2 / u^2$, or $s_\nu^2 = \left(u^2/\nu\right) x$, and compute the variance in s_ν^2.

$$\sigma^2\left(s_\nu^2\right) = \mathrm{var}\left(s_\nu^2\right) = \frac{u^4}{\nu^2}\mathrm{var}\left(x\right) . \tag{1}$$

For a χ^2-distributed variable *x*, we have

$$\mathrm{var}\left(x\right) = 2\nu \, ,$$

so that

$$\sigma^2\left(s_\nu^2\right) = \frac{2u^4}{\nu} \, , \tag{2}$$

and

$$\nu = 2\frac{u^4}{\sigma^2\left(s_\nu^2\right)} \, . \tag{3}$$

We now replace the sample variance s_ν^2 with the population variance u^2 and write

$$\nu \simeq 2\frac{u^4}{\sigma^2\left(u^2\right)} . \tag{4}$$

To obtain the variance $\sigma^2(u^2)$, we work with the expression for the uncertainty in a normally distributed error

$$u = \frac{L}{\varphi(p)} \, , \tag{5}$$

where $\pm L$ are error containment limits, *p* is the containment probability and

$$\varphi(p) = \Phi^{-1}\left(\frac{1+p}{2}\right) . \tag{6}$$

From Eq. (5), we have

$$u^2 = \frac{L^2}{\varphi^2(p)} \tag{7}$$

and the error in u^2 is

$$\varepsilon(u^2) \cong \left(\frac{\partial u^2}{\partial L}\right)\varepsilon(L) + \left(\frac{\partial u^2}{\partial p}\right)\varepsilon(p) \, . \tag{8}$$

Note that the variance in u^2 is synonymous with the variance in $\varepsilon(u^2)$. Hence

$$\begin{aligned}\sigma^2(u^2) &= \mathrm{var}(u^2) = \mathrm{var}[\varepsilon(u^2)] \\ &= \left(\frac{\partial u^2}{\partial L}\right)^2 \mathrm{var}[\varepsilon(L)] + \left(\frac{\partial u^2}{\partial p}\right)^2 \mathrm{var}[\varepsilon(p)] \\ &= \left(\frac{\partial u^2}{\partial L}\right)^2 u_L^2 + \left(\frac{\partial u^2}{\partial p}\right)^2 u_p^2 ,\end{aligned} \tag{9}$$

where $\varepsilon(L)$ and $\varepsilon(p)$ are assumed to be s-independent, and where u_L is the uncertainty in the containment limit L and u_p is the uncertainty in the containment probability p. In this expression, the equalities

$$u_L^2 = u_{\varepsilon_L}^2 = \mathrm{var}[\varepsilon(L)]$$

and

$$u_p^2 = u_{\varepsilon_p}^2 = \mathrm{var}[\varepsilon(p)]$$

were used.

It now remains to determine the partial derivates. From Eq. (5) we get

$$\left(\frac{\partial u^2}{\partial L}\right) = \frac{2L}{\varphi^2(p)} \tag{10}$$

and

$$\left(\frac{\partial u^2}{\partial p}\right) = -\frac{2L^2}{\varphi^3(p)} \frac{d\varphi}{dp} . \tag{11}$$

The derivative $d\varphi/dp$ is obtained easily. We first establish that

$$\begin{aligned}\frac{1+p}{2} &= \Phi[\varphi(p)] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\varphi(p)} e^{-\zeta^2/2}.\end{aligned}$$

Taking the derivative of both sides of this expression yields

$$\frac{1}{2} = \frac{1}{\sqrt{2\pi}} e^{-\varphi^2(p)/2} \frac{d\varphi}{dp} ,$$

from which we get

$$\frac{d\varphi}{dp} = \sqrt{\frac{\pi}{2}} e^{\varphi^2(p)/2} . \tag{12}$$

Substituting in Eq. (11) gives

$$\left(\frac{\partial u^2}{\partial p}\right) = -\frac{2L^2}{\varphi^3(p)} \sqrt{\frac{\pi}{2}} e^{\varphi^2(p)/2} . \tag{13}$$

Combining Eqs. (13) and (10) in Eq. (9), yields

$$\sigma^2(u^2) = \frac{4L^4}{\varphi^4(p)} \left[\frac{u_L^2}{L^2} + \frac{1}{\varphi^2(p)} \frac{\pi}{2} e^{\varphi^2(p)} u_p^2 \right] . \tag{14}$$

Substituting Eq. (14) in Eq. (4) and using Eq. (5) yields

$$\nu \simeq \frac{1}{2} \left[\frac{u_L^2}{L^2} + \frac{1}{\varphi^2(p)} \frac{\pi}{2} e^{\varphi^2(p)} u_p^2 \right]^{-1} . \tag{15}$$

Comparison with Eq. G3 of the GUM

Appendix G of the ISO Guide to the Expression of Uncertainty in Measurement (the GUM) provides an expression for the degrees of freedom for a Type B estimate

$$\nu \simeq \frac{1}{2}\frac{u^2}{\sigma^2(u)}\,. \tag{16}$$

From Eq. (5), we have

$$\begin{aligned}\varepsilon(u) &\simeq \left(\frac{\partial u}{\partial L}\right)\varepsilon(L)+\left(\frac{\partial u}{\partial p}\right)\varepsilon(p)\\ &= \frac{1}{\varphi(p)}\varepsilon(L)-\frac{L}{\varphi^2(p)}\frac{d\varphi}{dp}\varepsilon(p).\end{aligned} \tag{17}$$

Substituting from Eq. (12) gives

$$\varepsilon(u) \simeq \frac{1}{\varphi(p)}\varepsilon(L)-\frac{L}{\varphi^2(p)}\sqrt{\frac{\pi}{2}}\,e^{\varphi^2(p)/2}\varepsilon(p)\,.$$

Invoking the variance addition rule, we have

$$\begin{aligned}\sigma^2(u) &= \mathrm{var}(u) = \mathrm{var}[\varepsilon(u)]\\ &= \left(\frac{\partial u}{\partial L}\right)^2 \mathrm{var}[\varepsilon(L)]+\left(\frac{\partial u}{\partial p}\right)^2 \mathrm{var}[\varepsilon(p)]\\ &= \left(\frac{\partial u}{\partial L}\right)^2 u_L^2+\left(\frac{\partial u}{\partial p}\right)^2 u_p^2\\ &= \frac{1}{\varphi^2(p)}u_L^2+\frac{L^2}{\varphi^4(p)}\frac{\pi}{2}e^{\varphi^2(p)}\\ &= \frac{1}{\varphi^2(p)}\left[u_L^2+\frac{L^2}{\varphi^2(p)}\frac{\pi}{2}e^{\varphi^2(p)}\right].\end{aligned} \tag{18}$$

Substituting from Eqs. (5) and (18) in Eq. (16) gives

$$\nu \simeq \frac{1}{2}\left[\frac{u_L^2}{L^2}+\frac{1}{\varphi^2(p)}\frac{\pi}{2}e^{\varphi^2(p)}\right]^{-1}\,. \tag{19}$$

Comparison of Eq. (19) with Eq. (15) shows that using either Eq. (4) derived in this note or Eq. G3 of the GUM yields the same result.