Uncertainty Analysis and Parameter Tolerancing[1]

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Abstract

An uncertainty analysis methodology is described that is applicable to establishing and testing equipment parameter tolerances. The methodology develops descriptions of measurement uncertainty that relate directly to whether parameters will be acceptable for intended applications. An example is presented that illustrates the concepts involved.

Introduction

The tolerancing of equipment parameters is a multifaceted problem, which may involve competitive market pressures as well as technical considerations. In this paper, non-technical issues will not be discussed in detail. Instead, the focus will be on the technical concepts and methods involved in setting tolerance limits.

What are Tolerance Limits?

For the purposes of this paper, parameter tolerance limits are defined as limits that communicate to equipment users a range over which parameter values may be expected to be found with reasonable confidence. Parameters whose values are found within these limits are said to be in-tolerance. Parameters whose values fall outside these limits are said to be out-of-tolerance.

Tolerance Limit Criteria

The User's Perspective

In viewing a tolerance limit, a user or prospective user is faced with determining whether the range of values defined by the tolerance limits is acceptable for his or her intended application. In performing this evaluation, the following is tacitly assumed

- The limits will contain measurement errors.
- The containment probability is high.
- The time over which containment may be assumed is commensurate with the intended application.

From the user's perspective, then, there are a minimum of three variables that must be known:

1. The tolerance limits themselves.
2. The probability or confidence level that parameter values will be found within these limits.
3. The period of time over which this probability applies.

The Vendor's Perspective

The equipment manufacturer's technical objectives in establishing tolerance limits should be guided at least in part by customer's perceptions. This motivates the following two vendor criteria for parameter tolerance limits

- A low risk of rejection by the customer during receiving inspection.

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- A low risk of out-of-tolerance conditions perceived by the customer during periodic calibration or testing.

A third criterion, one that applies to the vendor, is motivated by economic considerations:

- A low false reject risk during product testing.

These criteria lead to the following management objectives with regard to equipment manufacturing and testing:

1. A low false accept risk during product testing.
2. A margin of safety compensating for uncertainty in the customer's calibration and/or testing process.
3. An acceptable uncertainty growth rate during usage.
4. A low false reject risk during product testing.

Meeting Tolerancing Criteria

Achieving the management objectives 1-4 above ensures that both vendor and customer objectives are met. Accordingly, objectives 1-4 will serve as the primary objectives of this paper. The tools for meeting these objectives are discussed in what follows. These tools emerge as a combination of uncertainty analysis methods, risk management methods and uncertainty growth projection methods.

Tolerancing Analysis Outline

The parameter tolerancing problem will be approached in three stages. In the first, the analysis of uncertainties in the production process is examined. In the second, the analysis of uncertainties and risks associated with product testing are discussed. In the third, the evaluation of uncertainty growth and customer perception are addressed.

At each of the stages, uncertainty estimates are made. These estimate are used to evaluate risks. In the production stage, uncertainty estimates are used to determine tolerance limits that provided a reasonable confidence that items rolling off the production line will be in-tolerance prior to testing.

In the product testing stage, uncertainty estimates are folded into risk analysis equations used to evaluate false accept risk and false reject risk during product testing.

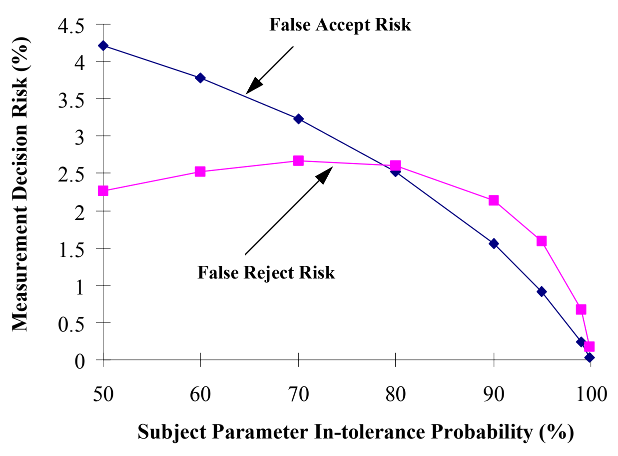

Figure 1. Risk vs. Parameter In-Tolerance Probability. Measurement decision risks for an accuracy ratio of 4:1 and a measuring parameter in-tolerance probability of 95%. False accept risk is computed from the user's perspective. Note that, for very low in-tolerance probabilities, false reject risk decreases because rejected parameters are likely to actually be out-of-tolerance. (Values computed using [3])

In the third stage, an uncertainty is estimated that reflects the in-tolerance probability of the product parameter during usage and also incorporates uncertainties that are likely to characterize the user's measurement system and measuring environment.

Production Process Uncertainty Analysis

For the most part, risks are reduced if the probability is high that parameter values to be tested lie within the tolerance limits. This is illustrated in Figure 1.

Obviously, the in-tolerance probability of parameters submitted for product testing plays a crucial role in the risk that in-tolerance parameters will be rejected and out-of-tolerance parameters will be accepted. Determining this in-tolerance probability involves evaluating the measurement uncertainties surrounding the production process.

Production Process Uncertainties

Production process uncertainties will be analyzed using the procedure depicted in Figure 2 [1, 2].

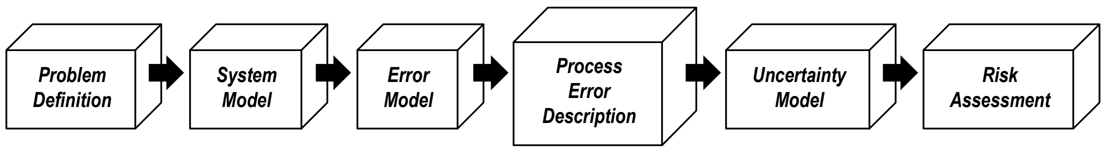

Figure 2. Uncertainty Analysis Procedure. The analysis of measurement uncertainty involves the use of a system model. The system model is used to derive the error model. The error model describes the influence that each source of measurement error has on the total uncertainty. Estimating uncertainties for each error source is required to determine the uncertainty model, from which the total uncertainty is computed. Once a total uncertainty estimated is obtained, an analysis of product testing and other risks can be made.

The specific steps in the procedure are

1. Define the quantity of interest. Determine what variables need to be measured.
2. Develop the system equation that describes the quantity in terms of measurable variables.
3. Develop an error model describing total measurement error as a function of source errors.
4. Identify process error components for each source. Estimate measurement process uncertainties.
5. Estimate the total uncertainty.
6. Evaluate risks and take appropriate action.

Problem Definition

The Quantity of Interest

The above process will be illustrated for a case where the item to be manufactured is a cylinder whose nominal value is to be 1 cc. The quantity of interest, or equipment parameter, for the present example is the cylinder's volume.

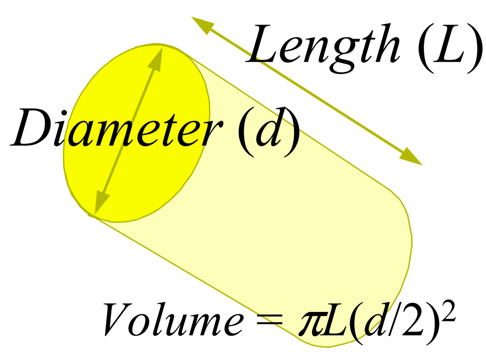

Figure 3. Sources of Error. The variables that are measured in determining the value of a subject parameter are sources of error. In the case of the volume of a cylinder, the sources of error are length measurement and diameter measurement.

System Model

The System Equation

We express the volume of the cylinder V in terms of the measurable variables length (L) and diameter (d)[2]

$$V = \pi L(d/2)^2. \qquad (1)$$

Eq.(1) is the *system equation* for the measurement.

Error Model

We recognize that each variable in the system equation is a potential source of error. Accordingly, we develop the error model by expanding Eq. (1) in a Taylor series [1, 2]. Ordinarily, this is done using partial derivatives. In the present example, we will use only high school algebra. In this approach, we write Eq. (1) as

[2]For simplicity, contributions due to surface irregularities will be ignored.

$$V_0 + \varepsilon(V) = \pi\left[L_0 + \varepsilon(L)\right]\left[\frac{d_0 + \varepsilon(d)]}{2}\right]^2, \tag{2}$$

where the notation $\varepsilon(\cdot)$ represents the error in the bracketed variable, and the zero subscripts indicate that the variables (error sources) are to be taken at their nominal (or "errorless") values. Multiplying out the terms in Eq. (2) gives

$$V_0 + \varepsilon(V) = \pi\left[L_0(d_0/2)^2 + \frac{1}{2}L_0 d_0 \varepsilon(d) + \frac{1}{4}L_0 \varepsilon^2(d) + (d_0/2)^2 \varepsilon(L) + (d/2)\varepsilon(L)\varepsilon(d) + \frac{1}{4}\varepsilon(L)\varepsilon^2(d)\right]$$

In cases where the errors in measurement are small relative to nominal values, the second and higher order terms may be dropped. In most applications this will be appropriate. Eq. (2) then becomes

$$V_0 + \varepsilon(V) = \pi L_0(d_0/2)^2 + \pi(d_0/2)^2 \varepsilon(L) + \frac{1}{2}\pi L_0 d_0 \varepsilon(d),$$

so that the error in volume is given by

$$\begin{aligned} \varepsilon(V) &= \pi(d_0/2)^2 \varepsilon(L) + \frac{1}{2}\pi L_0 d_0 \varepsilon(d) \\ &= c_L \varepsilon(L) + c_d \varepsilon(d). \end{aligned} \tag{3}$$

Eq. (3) is the error model for the determination of the volume V. The coefficients are

$$c_L = \pi(d_0/2)^2$$

and (4)

$$c_d = \frac{1}{2}\pi L_0 d_0 .$$

Process Error Description

The perceived (measured) values of the variables L and d in Eq. (1) are obtained in measurement processes. If each process is susceptible to some definable number of process error sources, then these error sources comprise components of $\varepsilon(L)$ and $\varepsilon(d)$.

Process Error Components

It has been found useful to break process error components down as follows [1, 2]:

Error Component	**Description**
Subject Parameter Bias	Systematic discrepancy between the "true" value and the nominal or reading value of a parameter being measured.
Measuring Parameter Bias	Systematic discrepancy between the "true" value and the nominal or reading value of a parameter performing a measurement.
Subject Parameter Random	Random fluctuations in the value of a parameter being measured.
Measuring Parameter Random	Random fluctuations in the value of a parameter performing a measurement.
Subject Parameter Resolution	Error due to the finite precision with which values of a parameter being measured can be perceived.
Measuring Parameter Resolution	Error due to the precision with which values of a parameter performing a measurement can be perceived.
Data Acquisition	Error due to acquiring data from measurements. Includes data sampling error, computation or "round off" error and operator bias.
Stress Response	Error due to stresses of shipping and handling of an item following measurement. Stress response error is important in cases where a measured parameter's value is reported externally and the measured item is physically moved from the measurement environment to another location.

Environment/Ancillary Equipment	Error due to environmental factors or to ancillary equipment, such as temperature monitoring devices.
Miscellaneous	Error due to sources peculiar to a given measurement scenario.

Assume that in the present analysis, we need to concern ourselves only with the process error components measuring parameter bias, measuring parameter random error, measuring parameter resolution error, data acquisition error (in the form of operator bias), and environmental factors. The expressions for the measurement errors in length and diameter are then

$$\varepsilon(L) = \varepsilon_{bias}(L) + \varepsilon_{ran}(L) + \varepsilon_{res}(L) + \varepsilon_{op}(L) + \varepsilon_{env}(L)$$

and (4)

$$\varepsilon(d) = \varepsilon_{bias}(d) + \varepsilon_{ran}(d) + \varepsilon_{res}(d) + \varepsilon_{op}(d) + \varepsilon_{env}(d) \ .$$

Uncertainty Model

So far, we have focused on how errors in volume are composed of errors in length and diameter measurement and how these errors, in turn, are composed of process error components. We now examine the question of how these errors relate to the uncertainty in the measurement of the volume V. To begin to answer this question, we first construct the uncertainty model. In doing this, we are guided by the following axiom [2]

> **Axiom:** The uncertainty in the measured value of a quantity is equal to the uncertainty in the measurement error for the quantity.

Given this axiom, we write the uncertainty in V as

$$u^2(V) = c_L^2 u^2(L) + c_d^2 u^2(d) + 2c_L c_d \rho_{Ld} u(L) u(d) \ , \tag{5}$$

where the quantity ρ_{Ld} is the correlation coefficient between L and d.

Component Uncertainties

Given the makeup of $\varepsilon(L)$ and $\varepsilon(d)$ in Eq. (4), we can safely assume that for each error source, the error components are statistically independent. This allows us to write the component uncertainties as

$$u^2(L) = u_{bias}^2(L) + u_{ran}^2(L) + u_{res}^2(L) + u_{op}^2(L) + u_{env}^2(L)$$

and (6)

$$u^2(d) = u_{bias}^2(d) + u_{ran}^2(d) + u_{res}^2(d) + u_{op}^2(d) + u_{env}^2(d)$$

Cross-Correlations

In some cases, the process error of one error source is correlated with the process error of another. These cases are marked by nonzero cross-correlations. The appropriate correlation coefficients can be readily derived using basic probability theory. The general expressions are given in [2].

For the present example, assume that the length and diameter measurements are made using the same device and that both measurements are made in the same environment by the same operator. Then there will be nonzero cross-correlations between length and diameter bias error, length and diameter operator error and length and diameter environmental error. All other correlations will be zero. These considerations yield the expression [2]

$$\rho_{Ld} = \frac{1}{u_L u_d}\left[\rho_{bias}(L,d) u_{bias}(L) u_{bias}(d) + \rho_{op}(L,d) u_{op}(L) u_{op}(d) + \rho_{env}(L,d) u_{env}(L) u_{env}(d)\right] \ . \tag{7}$$

Suppose for simplicity that we have

$$\rho_{bias}(L,d) = 1.0$$

$$\rho_{op}(L,d) = 0.5$$

$$\rho_{env}(L,d) = 1.0.$$

Substituting these values in Eq. (7) gives

$$\rho_{Ld} = \frac{1}{u(L)u(d)}\left[u_{bias}(L)u_{bias}(d) + \frac{1}{2}u_{op}(L)u_{op}(d) + u_{env}(L)u_{env}(d)\right]. \quad (8)$$

Uncertainty Combination

Using Eq. (8) in Eq. (5) gives

$$u^2(V) = c_L^2 u^2(L) + c_d^2 u^2(d) + 2c_L c_d\left[u_{bias}(L)u_{bias}(d) + \frac{1}{2}u_{op}(L)u_{op}(d) + u_{env}(L)u_{env}(d)\right]. \quad (9)$$

Eq. (9) is the total uncertainty in the determination of the volume V in terms of uncertainties in the measurement of length and diameter.[3] To better see the components of these contributions, we expand Eq. (9) by substituting from Eq. (6). This yields

$$\begin{aligned} u^2(V) = {} & \left[c_L u_{bias}(L) + c_d u_{bias}(d)\right]^2 + \left[c_L u_{env}(L) + c_d u_{env}(d)\right]^2 + \left[c_L^2 u_{op}^2(L) + c_L c_d u_{op}(L)u_{op}(d) + c_d^2 u_{op}^2(d)\right] \\ & + c_L^2 u_{ran}^2(L) + c_d^2 u_{ran}^2(d) + c_L^2 u_{res}^2(L) + c_d^2 u_{res}^2(d). \end{aligned} \quad (10)$$

At this point, we digress slightly to make an observation. To do this, we denote the uncertainty contribution from a given source with an upper case letter U. For instance, the contribution from length measurement bias would be $U_{bias}(L) = c_L u_{bias}(L)$; the contribution from operator bias in the diameter measurement would be $U_{op}(d) = c_d u_{op}(d)$, and so on. With this notation, Eq. (10) becomes

$$\begin{aligned} u^2(V) = {} & \left[U_{bias}(L) + U_{bias}(d)\right]^2 + \left[U_{env}(L) + U_{env}(d)\right]^2 + \left[U_{op}^2(L) + U_{op}(L)U_{op}(d) + U_{op}^2(d)\right] \\ & + U_{ran}^2(L) + U_{ran}^2(d) + U_{res}^2(L) + U_{res}^2(d). \end{aligned} \quad (11)$$

Equation (10) shows that, *in this example*, because of the correlation coefficients, the bias and environmental uncertainty contributions are each added linearly, the random and resolution uncertainty contributions are added in root-sum-square (rss) and the operator bias uncertainty contributions are added in a "composite" linear-rss manner.

This calls to mind debates that were at the forefront of uncertainty analysis technology a few years back as to whether uncertainties should be added in rss or summed linearly. From Eq. (11), we see that both sides of the debate are represented simply by taking correlation terms into account.

Test Process Uncertainty Analysis

To simplify matters, assume that the same measuring process is used during product testing as is involved in setting up the production process. If so, then all we need to do is employ the total measurement uncertainty in a set of risk equations. These equations are discussed in references [1] and [2].

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[3] In the production process, the uncertainties are manifested in the dimensions of templates, jugs, molds or other production artifacts.

We will shortcut this practice here by using an off-the-shelf software package to compute risks [3]. To use this package, we need to separate bias uncertainty from the other process uncertainty components. This is done as follows. From Eq. (11), we can write

$$\begin{aligned} u_{bias}(V) &= U_{bias}(L) + U_{bias}(d) \\ u_{ran}^2(V) &= U_{ran}^2(L) + U_{ran}^2(d) \\ u_{res}^2(V) &= U_{res}^2(L) + U_{res}^2(d) \\ u_{env}(V) &= U_{env}(L) + U_{env}(d) \\ u_{op}^2(V) &= U_{op}^2(L) + U_{op}(L)U_{op}(d) + U_{op}^2(d) \end{aligned} \tag{12a}$$

so that

$$\begin{aligned} u^2(V) &= u_{bias}^2(V) + u_{ran}^2(V) + u_{res}^2(V) + u_{op}^2(V) + u_{env}^2(V) \\ &= u_{bias}^2(V) + u_{other}^2(V) \end{aligned} \tag{12b}$$

where

$$u_{other}^2(V) = u_{ran}^2(V) + u_{res}^2(V) + u_{op}^2(V) + u_{env}^2(V) \ . \tag{12c}$$

As indicated, these expressions will be used in a risk analysis software package. We will return to them later.

Uncertainty Growth Analysis

During usage, the product parameter is subject to stresses that may be considered primarily random in type, magnitude and direction. For this reason, the uncertainty in the bias of a product parameter may grow with time since test or calibration. This has been found to be the case for a wide variety of measuring and test equipment [4, 5]. For the cylinder volume example considered in this paper, bias uncertainty growth is not likely to be a major concern in many applications by virtue of the fact that the volume is not an adjustable parameter. In some applications, however, if the cylinder is compressed or deformed during usage, the time-dependence of the bias uncertainty may be worth accounting for. For discussion purposes, we will assume that this is the case.

There are several ways to project uncertainty growth over time. A method using calibration interval analysis concepts has been employed in performing post-deployment analyses of uncertainty growth [6]. Another approach involves the use of life testing methods.

In life testing, a sample of items are selected randomly and measured periodically over a time frame that is assumed to adequately provide visibility of uncertainty growth. During the periods or intervals between measurements, the items are subjected to stresses of the kind expected to have an influence of parameter bias. In some life testing studies these stresses are elevated to levels that are somewhat higher than those expected to occur in practice. Such studies are referred to as "accelerated life testing" studies [7].

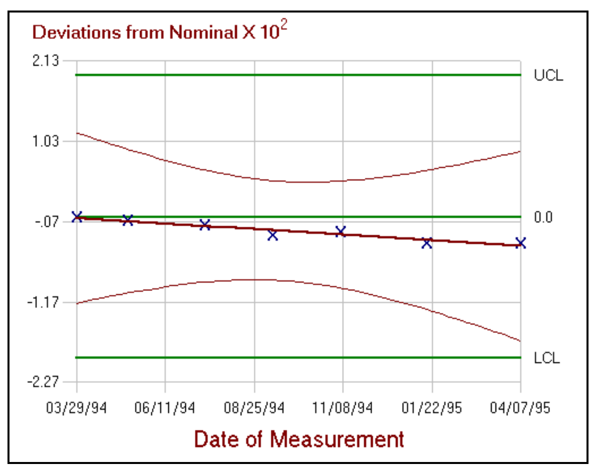

Figure 4. Life Testing Study for Uncertainty Growth Analysis. The cylinder volume is sampled over time to determine bias drift and bias drift uncertainty. (Chart developed using [6])

In this paper, we adopt a "control chart" life testing approach for modeling uncertainty growth vs. time. In this approach, successive measurements are shown as deviations from nominal and the upper and lower control limits are the product parameter tolerances (to be determined later). At each measurement, parameters may be adjusted to nominal or left alone — provided the adjustment practice is consistent throughout the study. Of course, with a non-adjustable parameter like cylinder volume, an "adjustment" consists of applying a correction factor.

The whole point of the life testing study is to arrive at some idea of expected uncertainty growth to be experienced during product use. If the user is expected to apply correction factors at successive tests or calibrations, then the same should be done in the study. If the user is not expected to apply correction factors, then the study should adopt the same practice.

Suppose that a life testing study is performed and the results are as shown in Figure 4. In figure 4, the measured values are plotted over the duration of the study and a linear fit to the data is achieved. The linear fit projects bias drift as a function of time with upper and lower curves surrounding the linear curve fit. These curves represent the uncertainty in the projected bias drift.

If the uncertainty due to the time-dependence of the product parameter bias is denoted $u_{growth}(t)$, then, since each measured point is susceptible to the measurement bias uncertainty in Eq. (12), the bias uncertainty in the parameter can be written

$$u_{bias}^2(V,t) = u_{bias}^2(V) + u_{growth}^2(t) \ . \tag{13}$$

Using the bias uncertainty $u_{bias}(V,t)$ in the risk analysis software, together with reasonable estimates of process uncertainty to be encountered in the user's measuring environment, results in a computation of false accept and false reject risk expected to be experienced by the user during testing or calibration.

Product Testing Risk Analysis

As shown in Figure 1, false accept and false reject risks during product testing are sensitive to the in-tolerance probability of product parameters prior to test. Suppose that we stipulate that we want at least 95% of untested cylinders to be within tolerance coming off the production line. We can derive test limits for the product using the expression

$$V_{tol} = t_{0.975,\nu}\, u(V) \ , \tag{14}$$

where $u(V)$ is given in Eqs. (10) and (11) and $t_{0.975,\nu}$ is the t-statistic for two-sided 95% confidence limits. The parameter ν is the "degrees of freedom" association with the uncertainty estimate $u(V)$ [1, 2, 8]. Suppose that, for the present example, we have

$$\begin{aligned} u(V) &= 0.018 \text{ cc} \\ \nu &= 190. \end{aligned} \tag{15}$$

Then, since $t_{0.975,190} \cong 1.97$, we get

$$V_{tol} = 0.036 \text{ cc} \ . \tag{16}$$

We are now ready to compute product testing risks. Assume that we test the product using the same measuring system as is used to establish the production process. Suppose that the variables and uncertainties involved are as shown in Table 1.[4] From the table, and from Eqs. (1), (4), (11) and (12), we determine the following

[4]Estimates obtained using [6].

$$V_0 = 1.0 \text{ cc}$$

$$c_L = 1.54 \text{ cm}^2$$

$$c_d = 1.43 \text{ cm}^2$$

$$u_{bias}(V) = 0.0134 \text{ cc}$$

$$u_{ran}(V) = 0.0075 \text{ cc}$$

$$u_{res}(V) = 0.0061 \text{ cc}$$

$$u_{op}(V) = 0.0062 \text{ cc}$$

$$u_{env}(V) = 0.000013 \text{ cc}$$

and

$$u_{other}(V) = 0.0115 \text{ cc} .$$

Table 1. Cylinder Uncertainty Analysis Variables

Variable	Value	Degrees of Freedom
L_0	0.65 cm	
d_0	1.40 cm	
$u_{bias}(L)$	0.0045 cm	∞
$u_{ran}(L)$	0.0029 cm	6
$u_{res}(L)$	0.0029 cm	∞
$u_{op}(L)$	0.003 cm	∞
$u_{env}(L)$	0.0000068 cm	∞
$u_{bias}(d)$	0.0045 cm	∞
$u_{ran}(d)$	0.0042 cm	6
$u_{res}(d)$	0.0029 cm	∞
$u_{op}(d)$	0.003 cm	∞
$u_{env}(d)$	0.0000015 cm	∞

It remains to compute an in-tolerance probability for the measurement bias relative to the tolerance limits $\pm V_{tol}$. This probability is obtained from

$$P_{bias} = 2\Phi\left[\frac{V_{tol}}{u_{bias}(V)}\right] - 1, \qquad (17)$$

where $\Phi(\cdot)$ is the normal distribution function.[5] P_{bias} is the probability that the limits $\pm V_{tol}$ will contain measurement biases encountered in product testing. Substituting the appropriate values in Eq. (17) gives

$$P_{bias} = 0.993 .$$

The foregoing numbers were entered in the software package mentioned earlier. The results are shown in Figures 5 and 6. In Figure 6, note the 1.0 cc ±0.036 cc tolerance limits for the subject unit (the cylinder volume under test) and the 0.036 cc tolerance offset for the MTE (measuring and test equipment) system. Note also, the 95% and 99.3% in-tolerance probabilities, respectively, for the subject unit and the measuring system.

From Figure 6, we see that testing the cylinder to ±0.036 cc results in an excessive false reject risk. We could lower the false reject risk by applying a guardband test limit that lies outside the ±0.036 limits. For example, as shown in Figures 7 and 8, setting a guardband of 1.375 times the tolerance limit equalizes false accept and false reject risks.

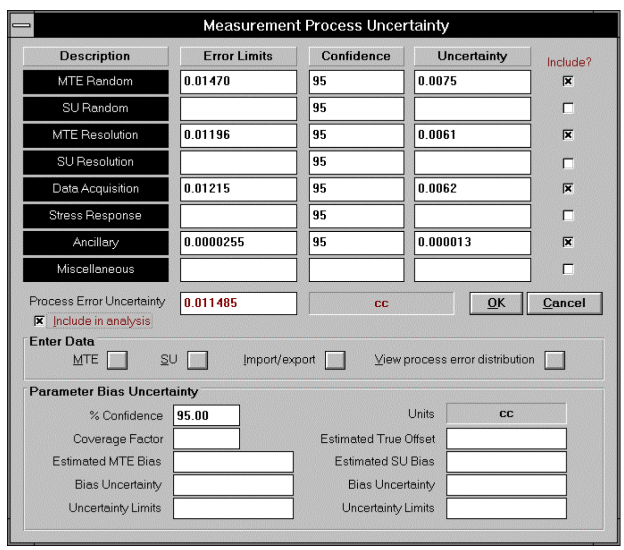

Figure 5. "Other" Process Uncertainty. The process uncertainty (excluding bias uncertainty) involved in product testing the cylinder volume. Operator bias uncertainty is covered under Data Acquisition while environmental uncertainty is covered under the Ancillary category.

[5]We use the normal distribution rather than the t-distribution. This approximation is justified because of the degrees of freedom involved.

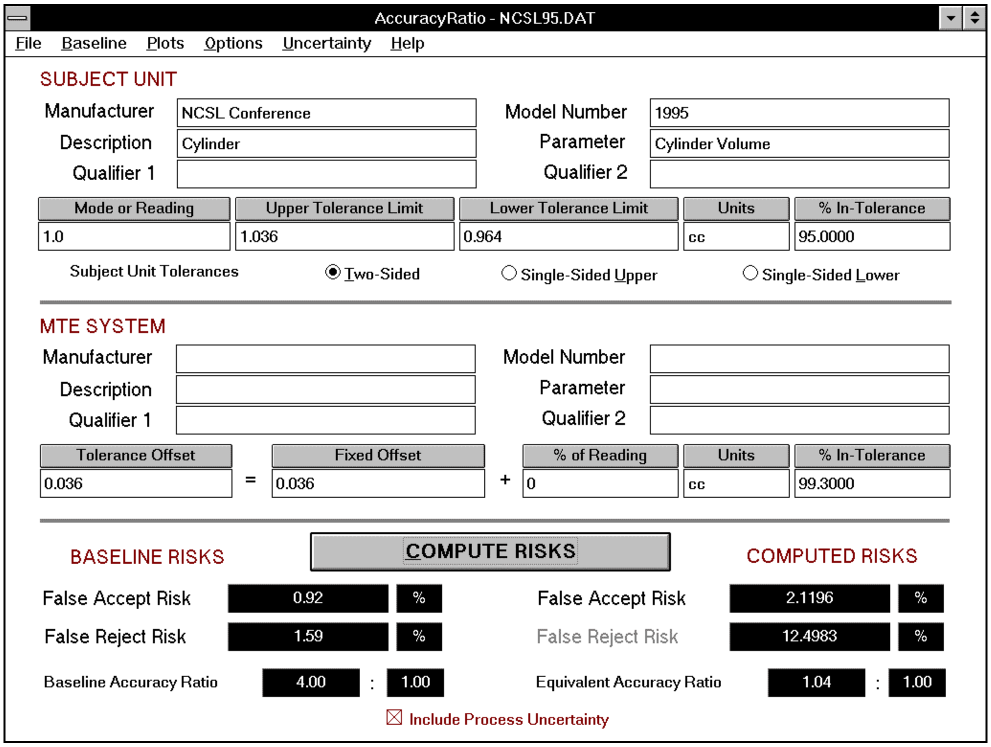

Figure 6. Product Testing Risk Analysis. False accept and false reject risks associated with product testing are inordinately high for product tolerances of ±0.036 cc.

As figure 7 shows, applying a guardband to reduce false reject risk has the effect of increasing false accept risk [3, 9, 10]. However, even without applying a guardband, the false accept risk of 2.1% shown in Figure 6 is somewhat "borderline" as it is. For this reason, applying a guardband is not really a viable solution in this case.

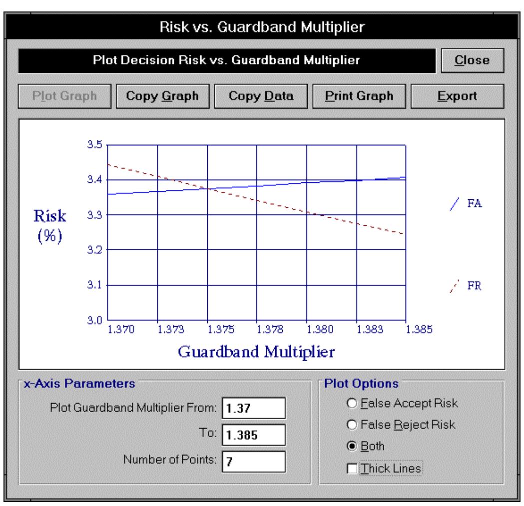

Figure 7. Guardband Analysis. By plotting false accept and false reject risks against guardband multiplier, an appropriate guardband can be found. A guardband that equalizes false accept and false reject risk is attractive in that the "true" percent in-tolerance matches the "observed" percent in-tolerance.

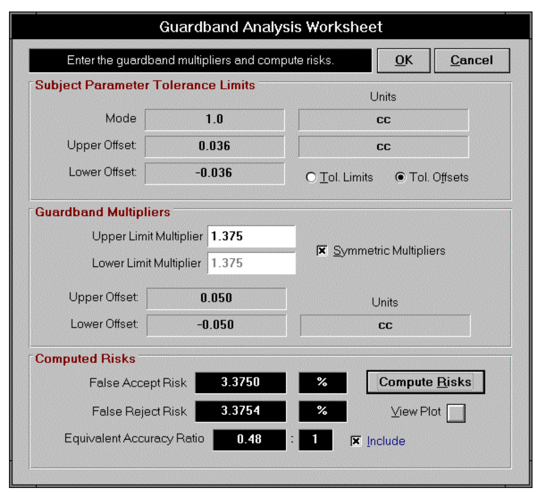

Figure 8. Guardband Results. A guardband of 1.375 serves to equalize false accept and false reject risk.

What we do instead is relax the tolerance limit on the cylinder volume to be more commensurate with the product testing capability. For instance, applying a tolerance limit of ±0.05 cc reduces false reject risk from 12.5% to 4.5% and reduces false accept risk from 2.2% to only 0.24%. These results are shown in Figure 7. As the figures shows, increasing the cylinder tolerance limit to ±0.05 cc, while holding everything else constant, increases the subject unit in-tolerance probability from 95% to 99.4%.

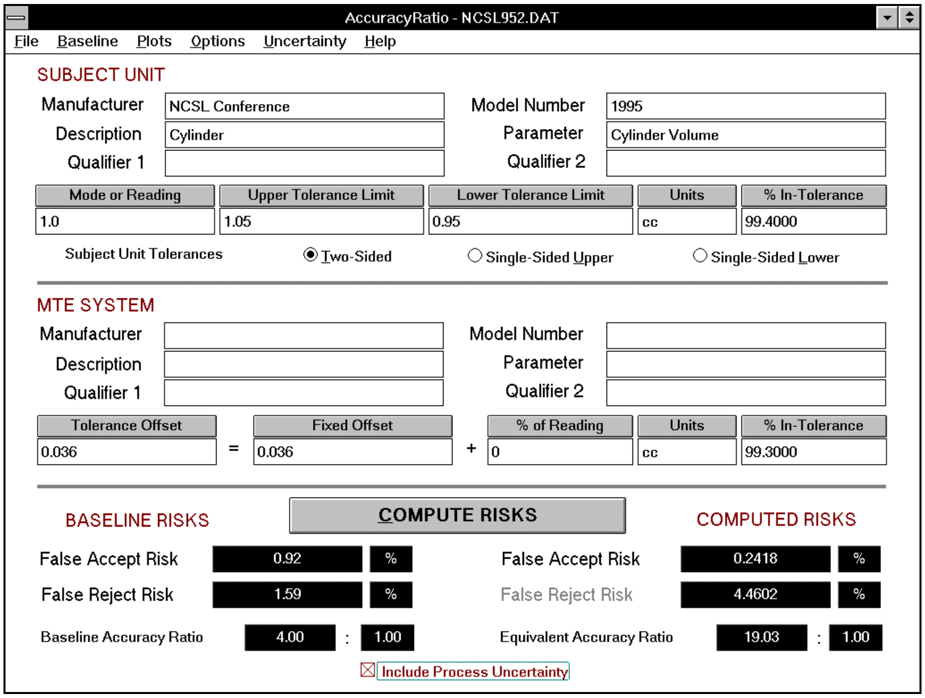

Figure 9. Revised Product Testing Risks. By changing the product tolerances from 0.036 cc to 0.05 cc, both false accept and false reject risks are reduced considerably.

While expanding the tolerance limits puts false reject risk on a more economically sound footing, the effect on false accept risk is perhaps even more beneficial. A false accept risk of only 0.24% means that virtually all cylinders will leave the factory in an in-tolerance condition.

User Testing/Calibration Risk Analysis

The user's perception of the quality of a toleranced item will be examined under two sets of circumstances. In the first, the product is evaluated in a receiving inspection process. In the second, the product is evaluated periodically at the end of its test or calibration interval.

Receiving Inspection

The product leaves the factory and is shipped to the user. During transport, stresses may occur that reduce the in-tolerance probability from the post-test value. While these stresses can be accounted for (see Figure 5), we will assume that the tested cylinders are packed in such a way that shipping and handling stresses are not a factor in the present analysis.

With this assumption in mind, we state that products arrive at the user's facility with only 0.24% out-of-tolerance due to false accept risk during product testing. This means that 99.76% of the cylinders will be in-tolerance as received by the user.

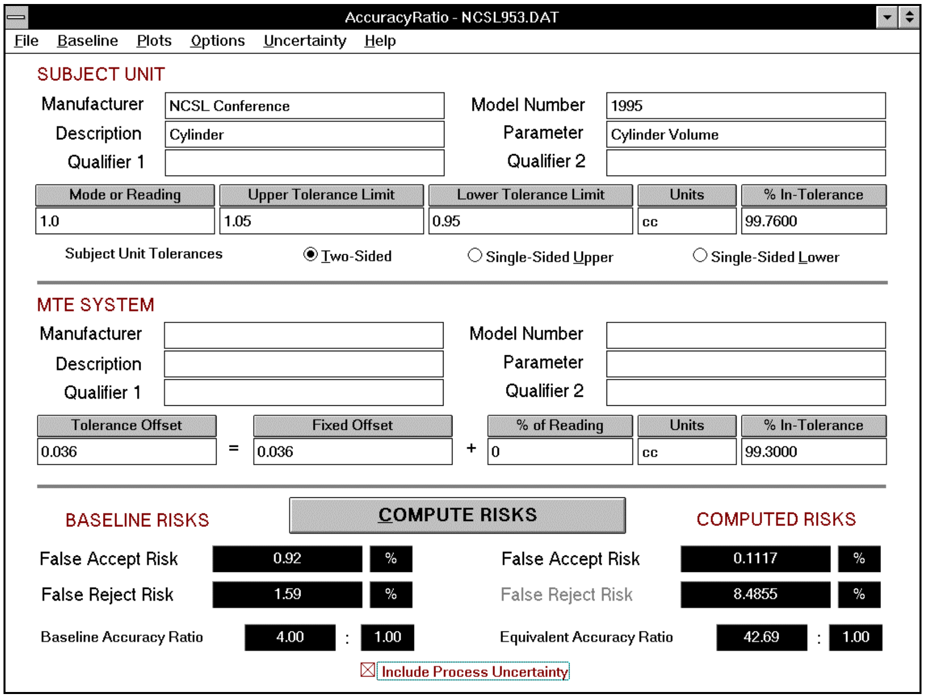

Figure 10. Receiving Inspection Risks. False accept and false reject risks computed for the user's receiving inspection. Figures are based on an assumed test system accuracy of ±0.036 cc with a test process uncertainty of 0.02 cc (excluding bias).

We now attempt to determine what the user's perception of delivered product will be. To do this, we first need to make some assumptions about the uncertainties surrounding the user's test system. These assumptions may be based on a knowledge of the user's facility or on reasonable assumptions concerning user facilities in general.

Suppose that we surmise that the user's test capability is equivalent to the product testing capability shown in Figure 6 and that the user's test process uncertainty is approximately 0.02 cc (excluding bias). Entering these numbers in the risk analysis software package, we get the results shown in Figure 10.

As Figure 10 shows, since there is both a high probability of in-tolerance product and a low test accuracy relative to product tolerances, the user will falsely reject approximately 8.5% of delivered items. This is highly undesirable. The situation can be rectified in at least two ways.

One way is to further relax the product specification. For instance, if the tolerance limits are expanded to ±0.10 cc, the product testing false accept and false reject risks drop to 0.18% and 0.92%, respectively and the user's receiving inspection false accept and false reject risks drop to 670 ppm and 1.14%.

While these numbers are excellent, issuing a product with such expanded tolerances may not be advisable from a competitive market standpoint. An alternative would be to inform the user that, if test uncertainty is high, relative

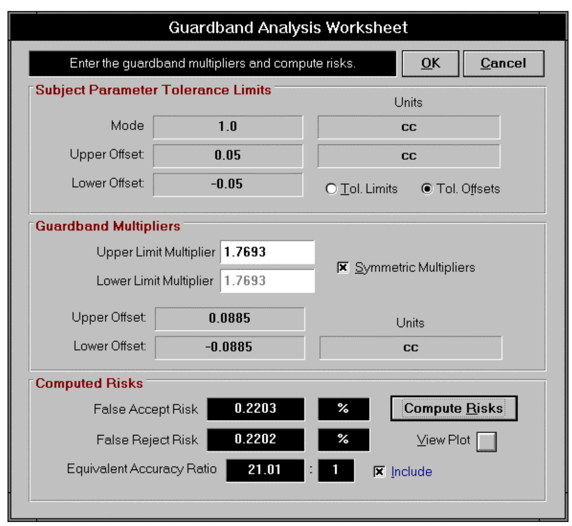

Figure 11. Receiving Inspection Guardband. A suggested guardband multiplier to be used during receiving inspection. The multiplier equalizes false accept and false reject risk while holding each to a negligible value.

to product tolerances, a high false reject risk will be encountered. The situation can be remedied by suggesting that the user employ a guardband test limit that lies outside the product tolerance limits.[6] Figure 11 shows an example of an appropriate guardband multiplier.

Periodic Inspection - Accommodating Uncertainty Growth

Imagine that, when the delivered items are placed in use, they will be tested or calibrated periodically. Since stresses will be encountered in use, an uncertainty growth term must be included in the bias uncertainty part of the measurement process uncertainty. The relevant expression is Eq. (13).

Suppose that we perform a life testing study, as shown in Figure 4, and that the study yields the following results:

Recommended Test/Calibration Interval	= 6 months
Bias Drift	= - 0.0035 cc ± 0.0013 cc (at 6 months)

Given these figures, the bias uncertainty growth at the time of test or calibration is[7]

$$u_{growth}^2(t) = (0.0035 \text{ cc})^2 + (0.0013 \text{ cc})^2$$

$$= (0.0037 \text{ cc})^2 ,$$

The bias uncertainty at the time of test or calibration is given by

$$u_{bias}^2(V,t) = u^2(V) + u_{growth}^2(t)$$

The variable $u(V)$ is the uncertainty at the beginning of the test or calibration interval. Recalling that we computed a user false accept risk of around 0.22% (assuming the recommended guardbands are employed), the figure for $u(V)$ is approximately[8]

$$u(V) \cong \frac{V_{tol}}{\Phi^{-1}\left[\frac{1+P_{bias}}{2}\right]} = \frac{0.05 \text{ cc}}{\Phi^{-1}\left(\frac{1+0.9982}{2}\right)}$$

$$= (0.05/3.06) \text{ cc} \cong 0.016 \text{ cc} .$$

With this value for $u(V)$, the bias uncertainty at the time of test or calibration becomes

$$u_{bias}^2(V,t) = (0.016 \text{ cc})^2 + (0.0037 \text{ cc})^2$$

$$= (0.0164 \text{ cc})^2 .$$

This bias uncertainty, in turn, corresponds to an in-tolerance probability (relative to the tolerance limits of ±0.05 cc) of 99.77%. Employing this number, along with the u_{other} value of 0.02 cc estimated earlier, yields the results shown in Figures 12 and 13.

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[6]Of course, the point must be clearly, albeit diplomatically, made that this is required because of testing uncertainties rather than because of a shoddy product.

[7]Including the bias drift as an uncertainty term in the manner shown here is somewhat crude. A more rigorous approach would involve correcting for bias drift systematically, leaving only the uncertainty in the bias projection.

[8]We have no data on the user's degrees of freedom. Accordingly, we employ the normal distribution rather than the *t*-distribution. Since the estimates we are using are approximate, the lack of refinement introduced by this practice is negligible.

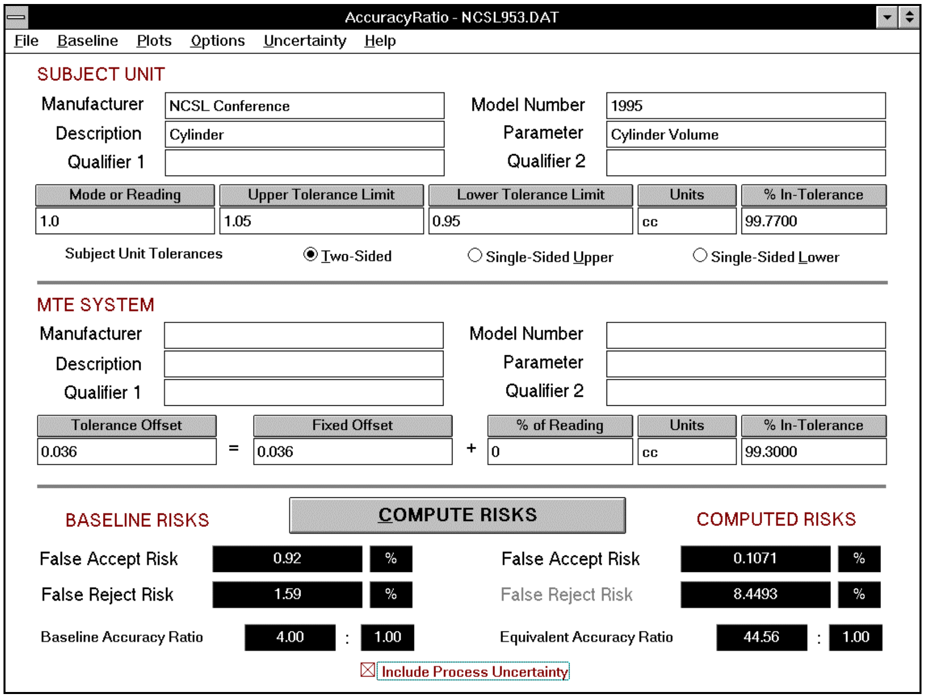

Figure 12. Periodic Inspection Risks. False accept and false reject risks during periodic test or calibration.

Figure 12 shows that, during test or calibration, false accept risk is low, but false reject risk is again high. This means that, although the cylinder volumes of nearly all products tested or calibrated will be in-tolerance, over 8% will be perceived as being out-of-tolerance. Again, we could remedy the situation either by expanding the tolerance limits or by recommending the use of the same guardband multipliers that were suggested for receiving inspection. If the latter course if followed, then the risks are as shown in Figure 13.

From Figure 13, we see that it can be argued that, provided the user employs suitable guardbands, the tolerance limits of ±0.05 cc lead to a favorable user perception of product quality. From the earlier discussions on product testing and testing at receiving inspection, we are justified in concluding that these limits are appropriate for the product.

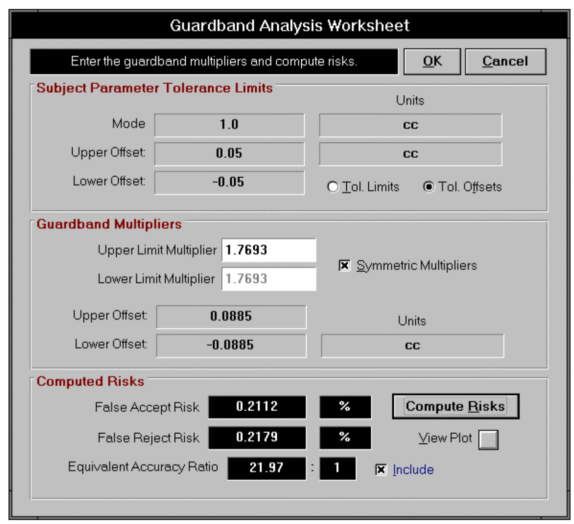

Figure 13. Periodic Inspection Guardbands. Using the same guardband multipliers as were employed during receiving inspection leads to an accurate perception of product in-tolerance probability at the time of test or calibration.

Conclusion

The rigorous technical evaluation of product tolerances requires the employment of uncertainty analysis and risk analysis methodologies. In employing these methodologies, a variety of considerations apply. These considerations are ensured by adhering to the following procedure

1. Develop an appropriate system equation and accompanying error model.
2. Determine relative contributions of error sources to total uncertainty.

3. Decompose error sources into process error components.
4. Account for correlations between terms.
5. Account for both bias uncertainty and other uncertainty in product testing.
6. Evaluate product testing false accept and false reject risks relative to product tolerances. Fine tune product tolerances to achieve reasonable risk levels.
7. Evaluate user perception of product quality during receiving inspection. If necessary, either modify product tolerances or suggest test guardbands or other compensating measures (e.g., the use of more accurate test systems).
8. Evaluate uncertainty growth and user perception of product quality at periodic test or calibration. If necessary, modify tolerances or suggest compensating measures.

The example presented in this paper illustrates this procedure. The example, while hypothetical, exhibits the important technical milestones in developing product tolerances. Unfortunately, not all of the mathematical or other supporting derivations could be included. These derivations are given in the references cited.

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