ESTIMATING CATEGORY B DEGREES OF FREEDOM[1]

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Abstract *- A method is presented for estimating uncertainties in cases where samples of data are unavailable. The method includes a formalism that provides a structure for extracting information from the measurement experience of scientific or technical personnel. This information is used to both estimate uncertainties and to approximate the degrees of freedom of the estimate. Using these results, confidence limits are developed that obviate the need for arbitrary coverage factors and misleading expanded uncertainties.*

INTRODUCTION

Historically, the uncertainty in a measured quantity has been equated with the standard deviation of the population of values that the quantity can assume. In early work on the subject, estimates of this uncertainty had been obtained by computing the standard deviations of samples of measurements. In these computations, the degrees of freedom variable is just the sample size minus one. Roughly speaking, this variable represents the amount of information on which the standard deviation estimate is based.

In the course of the development of *The Guide to the Expression of Uncertainty in Measurement* (the "GUM") [1], it was recognized that sample standard deviation estimates applied only to random variations experienced during measurement and did not take into account additional uncertainties, such as the uncertainty in the bias of the measuring parameter, the resolution of the measurement, possible operator bias, and errors due to environmental factors. Unfortunately, uncertainties due to these error sources could rarely be estimated by sample standard deviations, since samples of data from which to obtain these estimates were unavailable.

The conclusion of the authors of the GUM was that uncertainty estimates in the absence of sampled data were to be drawn from the scientific or technical experience of the analyst. Such estimates were labeled "Category B" to distinguish them from "Category A" estimates obtained from sampled data.

Beyond this, a practical methodology for obtaining Category B estimates was not provided. In the years following the publication of the first edition of the GUM, methods were developed that computed Category B standard deviations from *error containment limits* (e.g., parameter tolerance limits) and *containment probabilities* (e.g., in-tolerance probabilities).[2] These methods are applicable to errors arising from non-normal as well as normal distributions and have been incorporated in commercially available software.[3]

Until recently, regardless of the method used to obtain a Category B estimate, the estimate could not be rigorously applied as a statistic in computing confidence intervals or in statistical hypothesis testing. The principal reason for this is because, although an estimate of a population standard deviation could be computed, the accompanying number of degrees of freedom was not readily forthcoming. Without the degrees of freedom, appropriate t-statistics, commensurate with specified confidence levels, could not be applied, and confidence limits could not be developed.

What emerged instead were two artifices, the *coverage factor* and the *expanded uncertainty*. The former is used in place of the t-statistic, and the latter consists of an uncertainty estimate multiplied by the coverage factor. It is usually offered as an approximate confidence limit.[4]

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[1] Presented at the 2000 Measurement Science Conference, Anaheim, January 21, 2000.

[2] See, for example, References [10] - [12].

[3] Several software applications have been developed. An example is provided by Reference [8].

[4] A coverage factor of two is often employed. This value is usually felt to yield approximate 95% confidence limits. In practice, the actual confidence level is often very different from this value.

To make a long story short, using non-statistical coverage factors to obtain expanded uncertainties is not equivalent to applying t-statistics to obtain confidence limits. In the author's experience, using nominal coverage factors and applying the term "expanded uncertainty" have led to incorrect inferences and expensive miscommunications. Clearly, what is needed is a methodology that allows us to obtain Category B uncertainty estimates and degrees of freedom in such a way that we can return to the development of confidence limits and consign expanded uncertainty to the scrap heap. This paper is offered as a step toward developing this methodology.

CATEGORY B ESTIMATES

Typically, a Category B estimate of uncertainty emerges as a cognitive impression based on the recollected experience of a technical expert. In the current paradigm, all that is hoped for is an estimate of uncertainty without accompanying degrees of freedom or other statistics. In the absence of sampled data from which to determine the degrees of freedom associated with an estimate, the number of degrees of freedom is customarily taken to be infinite.

This practice in setting the number of degrees of freedom for a Category B estimate compromises its use as a statistic in hypothesis testing or in setting confidence limits. We *know* that the estimate is not based on an "infinite" amount of knowledge. In fact, we usually acknowledge that a Category B estimate is made from less complete knowledge than what typically accompanies a Category A estimate, which is characterized by a finite degrees of freedom. So, the upshot is that the estimates in which we often have the least confidence are treated as if we complete confidence in their values. The problem is exacerbated when attempting to use Welch-Satterthwaite [1, 2] or other means of computing the degrees of freedom for combined Category B and Category A estimates. In these computations, when we set the Category B degrees of freedom to infinity, the estimates about which we know the least tend to dominate the end result.

To compensate for the unavailability of rigorous degrees of freedom estimates, an "engineering" solution has been instituted that gives up on the whole idea of determining useful confidence intervals for Category B or mixed Category A,B estimates. In this practice, Category B estimates and mixed Category A,B estimates are uniformly multiplied by a fixed coverage factor that, hopefully, yields limits that bear some resemblance to confidence limits. In some cases, this practice may produce useful limits, but there is often no way to tell. Unfortunately, all that can truthfully be said about the practice is that, at one point we have an uncertainty estimate and at another point we have *k* times this estimate.[5] Obviously, we have added nothing to our knowledge or to the utility of the estimate by applying a fixed coverage factor.

What is needed for Category B estimates, is some way to draw from the experience of the estimator both the estimate itself and an accompanying degrees of freedom. It might be pointed out additionally that what is also needed is a means of determining the underlying statistical distribution for the estimate. However, such determinations are rarely made even for Category A estimates obtained from random samples. The usual assumption, which has considerable merit, is to assume an underlying normal distribution [3-7], which leads to the application of the Student's t distribution in computing confidence intervals. In this paper we will do likewise with Category B uncertainty estimates.

The approach to be taken is appropriate for the kind of uncertainty-related information that is available to technical experts. This approach begins by formalizing the Category B estimation thought process. This is done by viewing the process as an "experiment" involving independent Bernoulli trials.

BERNOULLI TRIALS AND CONTAINMENT PROBABILITY

Suppose we want to find the uncertainty in a variable y from independent Bernoulli trials that each determine (measure) whether the value of y lies within limits $\pm A$. The limits $\pm A$ will be referred to herein as *containment limits*.

We define the likelihood function for the *ith* trial of n independent trials in the usual way:

$$L_i = p^{x_i}(1-p)^{1-x_i}$$

where

$$x_i = \begin{cases} 1, \text{ if } y_i \in \pm A \\ 0, \text{ otherwise,} \end{cases}, \; i = 1,2, \cdots, n,$$

and, where p is the probability that y is contained within A. The probability p is referred to as the *containment probability*.

A likelihood function is constructed from the results of the n trials according to

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[5] k is most often set equal to two.

$$L = \prod_{i=1}^{n} L_i = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} \ ,$$

from whence

$$\ln L = \sum_{i=1}^{n} x_i \ln p + \sum_{i=1}^{n} (1-x_i)\ln(1-p) \ .$$

The containment probability p is estimated by maximizing the likelihood function. This is done by setting the derivative of ln$L$ with respect to p equal to zero

$$\begin{aligned} \frac{\partial}{\partial p}\ln L &= \sum_{i=1}^{n}\frac{x_i}{p} - \sum_{i=1}^{n}\frac{(1-x_i)}{(1-p)} \\ &= \sum_{i=1}^{n}\frac{x_i - p}{p(1-p)} \qquad (1) \\ &= 0 \ . \end{aligned}$$

This yields an estimate for p of

$$p = \frac{\sum_{i=1}^{n} x_i}{n} \ , \qquad (2)$$

as expected.

The summation in Eq. (2) is the total number of trials measured or observed to lie within $\pm A$. We denote this quantity x:

$$x = \sum_{i=1}^{n} x_i \ ,$$

and write Eq. (2) as

$$p = \frac{x}{n} \ . \qquad (3)$$

ESTIMATING CATEGORY B UNCERTAINTY

If we assume a distribution for the variable y, then Eq. (3) allows us to estimate the uncertainty in y, based on n observations with outcomes $x_1, x_2, \ldots , x_n$. For the present discussion, we will assume that y is normally distributed with zero mean and standard deviation u_y. Then the uncertainty in y is determined from the containment limits $\pm A$ and the containment probability p according to

$$\begin{aligned} p &= \frac{1}{\sqrt{2\pi}u_y}\int_{-A}^{A} e^{-y^2/2u_y^2}dy \\ &= 2\Phi(A/u_y) - 1, \end{aligned}$$

so that

$$u_y = \frac{A}{\Phi^{-1}\left[(1+p)/2\right]} \ , \qquad (4)$$

where $\Phi(\cdot)$ is the normal distribution function and $\Phi^{-1}(\cdot)$ is the inverse function.[6] Substituting from Eq. (3) yields a "sample" standard deviation

$$s_y = \frac{A}{\Phi^{-1}\left[(1+x/n)/2\right]} \ . \qquad (5)$$

As stated earlier, we will take an approach to estimating Category B uncertainties that relates to the kind of information that is normally available to technical personnel with measurement expertise, i.e., technicians or engineers. Ordinarily, technicians or engineers do not respond sensibly to questions like "in your experience, what is the uncertainty in y?" Instead, they tend to express their knowledge of such uncertainties in statements like "out of n observations on the variable y, approximately x have been found to lie within $\pm A$;" or "y lies between $\pm A$ in approximately x out of n cases;" or "y lies between $\pm A$ about $100 \times p$ percent of the time;" etc. From our earlier discussion on Bernoulli trials, we see that such proclamations can be viewed as statements of the results of informal experiments involving Bernoulli trials.

Responses of the "x out of n" variety can form the basis for estimation of uncertainty using Eq. (4). If Bernoulli trials are systematically observed and recorded, such estimates may be regarded as Category A.[7] If, on the other hand, Bernoulli trials consist of informally recollected impressions based on experience, then the estimates are Category B. In both cases, it is possible to determine workable estimates of the degrees of freedom.

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[6] Tabulated values of $\Phi(\cdot)$ and $\Phi^{-1}(\cdot)$ can be found in most statistics textbooks.

[7] There are some that may remark that, due to the *a priori* assumption of a particular distribution used to compute σ in Eq. (4), that the estimate is Category B rather than Category A. If so, then virtually all estimates are Category B, in that we typically assume one distribution or another for uncertainty combination purposes. We also make *a priori* assumptions about the distributions in computing risks and other economic variables.

Category B Degrees of Freedom

In Appendix G of the GUM, a relation is given for calculating the Category B degrees of freedom for variables that are normally distributed:

$$\nu_B \cong \frac{1}{2}\frac{u_B^2}{\sigma^2(u_B)} \ , \qquad (6)$$

where u_B is a Category B uncertainty estimate, and $\sigma^2(u_B)$ is the variance in this estimate. In applying Eq. (6), the trick is to estimate $\sigma^2(u_B)$. For this, the GUM is not much help. The primary guidance is offered in an example where a value for $\sigma(u_B)$ is already assumed.

The lack of a methodology for estimating the variance in u_B is at the core of our difficulty in placing Category B estimates on a statistical footing. The development of such a methodology is described below.

Computation of the Variance in the Uncertainty

We generalize Eq. (4) to read

$$u_B = \frac{A}{\varphi(p)} \qquad (7)$$

where p is the containment probability, and

$$\varphi(p) = \Phi^{-1}\left[(1+p)/2\right] \ . \qquad (8)$$

The error in u_B due to errors in A and φ is obtained from Eq. (7) in the usual way

$$\begin{aligned}\varepsilon(u_B) &= \left(\frac{\partial u_B}{\partial A}\right)\varepsilon(A) + \left(\frac{\partial u_B}{\partial p}\right)\varepsilon(p) \\ &= \left(\frac{\partial u_B}{\partial A}\right)\varepsilon(A) + \left(\frac{\partial u_B}{\partial \varphi}\right)\frac{d\varphi}{dp}\varepsilon(p) \qquad (9) \\ &= \frac{\varepsilon(A)}{\varphi} - \frac{A}{\varphi^2}\frac{d\varphi}{dp}\varepsilon(p) \ ,\end{aligned}$$

where $\varepsilon(A)$ and $\varepsilon(p)$ are errors in A and p, respectively. Assuming statistical independence between these errors, the variance in u_B follows directly:

$$\begin{aligned}\sigma^2(u_B) &= \mathrm{var}[\varepsilon(u_B)] \\ &= \frac{u_A^2}{\varphi^2} + \frac{A^2}{\varphi^4}\left(\frac{d\varphi}{dp}\right)^2 u_p^2 \ . \end{aligned} \qquad (10)$$

Dividing Eq. (10) by the square of Eq. (7), we get

$$\frac{\sigma^2(u_B)}{u_B^2} = \frac{u_A^2}{A^2} + \frac{1}{\varphi^2}\left(\frac{d\varphi}{dp}\right)^2 u_p^2 \ . \qquad (11)$$

The derivative in Eq. (11) is obtained from Eq. (8). We first establish that

$$\begin{aligned}\frac{1+p}{2} &= \Phi(\varphi) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\varphi} e^{-\zeta^2/2} d\zeta \ .\end{aligned}$$

We next take the derivative of both sides of this equation with respect to p to get

$$\frac{1}{2} = \frac{1}{\sqrt{2\pi}} e^{-\varphi^2/2}\frac{d\varphi}{dp} \ ,$$

and, finally,

$$\frac{d\varphi}{dp} = \sqrt{\frac{\pi}{2}}\, e^{\varphi^2/2} \ . \qquad (12)$$

Substituting Eq. (12) in Eq. (11) yields

$$\frac{\sigma^2(u_B)}{u_B^2} = \frac{u_A^2}{A^2} + \frac{1}{\varphi^2}\frac{\pi}{2} e^{\varphi^2} u_p^2 \ . \qquad (13)$$

Application Formats

In applying Eq. (13), we are immediately confronted with the problem of obtaining u_A and u_p. As we will see, it turns out that these quantities can be estimated using a simple prescription. First, however, we must extract technical information from the analyst. To do this, we utilize two formats:

Format 1: Approximately X% ($\pm\Delta X$%) of observed values have been found to lie within the limits $\pm A$ ($\pm\Delta A$).

Format 2: Approximately x out of n values have been found to lie within the limits $\pm A$ ($\pm\Delta A$).

Use of Format 1

In using Format 1, the containment probability is given by

$$p = \frac{X}{100} \ , \qquad (14)$$

where X is the percentage of values of y observed within $\pm A$.

With Format 1, a technical expert is asked to provide $\pm$ limits for both the containment limits and the containment probability. These limits are used to estimate u_A and u_p. If we assume that the errors in the estimates of A and p are approximately uniformly distributed within $\pm\Delta A$ and $\pm\Delta p = \pm\Delta X\% / 100$, respectively, then we can write[8]

$$u_A^2 = \frac{(\Delta A)^2}{3}, \text{ and } u_p^2 = \frac{(\Delta p)^2}{3} . \qquad (15)$$

Substitution in Eq. (13) gives

$$\frac{\sigma^2(u_B)}{u_B^2} = \frac{(\Delta A)^2}{3A^2} + \frac{1}{\varphi^2}\frac{\pi}{2} e^{\varphi^2} \frac{(\Delta p)^2}{3} . \qquad (16)$$

Use of Eq. (16) in Eq. (6) yields an estimate for the category B degrees of freedom for Format 1:

$$\begin{aligned} \nu_B &\cong \frac{1}{2}\left(\frac{\sigma^2(u_B)}{u_B^2}\right)^{-1} \\ &\cong \frac{3\varphi^2 A^2}{2\varphi^2(\Delta A)^2 + \pi A^2 e^{\varphi^2}(\Delta p)^2} . \end{aligned} \qquad (17)$$

Note that, if ΔA and Δp are set to zero, then $\nu_B \to \infty$.

Use of Format 2

With Format 2, the variance in A is obtained as in Format 1. The variance in the containment probability p can be obtained by taking advantage of the binomial character of p:

$$u_p^2 = \frac{p(1-p)}{n} . \qquad (18)$$

Substitution in Eq. (13) and using Eq. (15) for the uncertainty in A yields

$$\frac{\sigma^2(u_B)}{u_B^2} = \frac{(\Delta A)^2}{3A^2} + \frac{1}{\varphi^2}\frac{\pi}{2} e^{\varphi^2} \frac{p(1-p)}{n} , \qquad (19)$$

and

$$\begin{aligned} \nu_B &\cong \frac{1}{2}\frac{u_B^2}{\sigma^2(u_B)} \\ &\cong \frac{3\varphi^2 A^2}{2\varphi^2(\Delta A)^2 + 3\pi A^2 e^{\varphi^2} p(1-p)/n} . \end{aligned} \qquad (20)$$

Note that, in cases where $p = 1$ or $p = 0$, we have $p(1-p) = 0$. If ΔA is also zero, then $\nu_B \to \infty$.[9]

[8]Use of the uniform distribution is appropriate here, since the ranges ΔA and Δp can be considered analogous to "limits of resolution," for which the uniform distribution is applicable. This obviates the need for estimating confidence levels for ΔA and Δp. Any lack of rigor introduced by this tactic is felt as a third order effect and does not materially compromise the rigor of our final result.

Obviously, where appropriate, we want to avoid cases where $\nu_B \to \infty$. It therefore behooves us to attempt to apply whatever means we have at our disposal to obtain a sensible estimate for p. The following examples illustrate the development and use of such estimates.

Examples

Formats 1 and 2 are incorporated in a commercially available software package [8] and in a freeware application [9]. They are exemplified in the figures below, taken from the freeware application. The third figure presents a restatement of Format 2 that may be easier to use in certain circumstances.

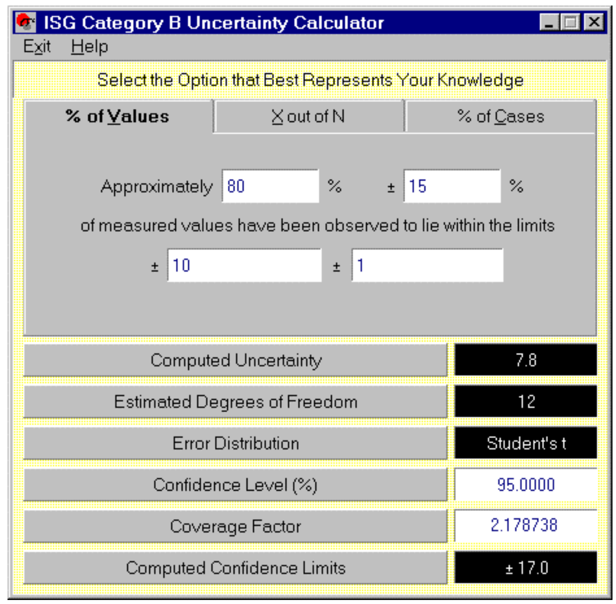

Figure 1. Degrees of freedom estimate for a case where approximately 80% of values are observed as being within the limits ±10. The approximate nature of the estimate is embodied in the secondary limits of ±15% and ±1. The format shown is Format 1.

[9] Actually the function $exp(\varphi^2)$ goes to infinity faster than $p(1-p)$ goes to zero. In these cases, it is more appropriate to use Eq. (10) directly and write

$$\sigma^2(u_B) = \frac{(\Delta A)^2}{3\varphi^2} + \frac{A^2}{\varphi^4}\left(\frac{d\varphi}{dp}\right)^2 \frac{p(1-p)}{n} ,$$

holding $\frac{1}{\varphi^4}\left(\frac{\partial\varphi}{\partial p}\right)^2$ to be finite.

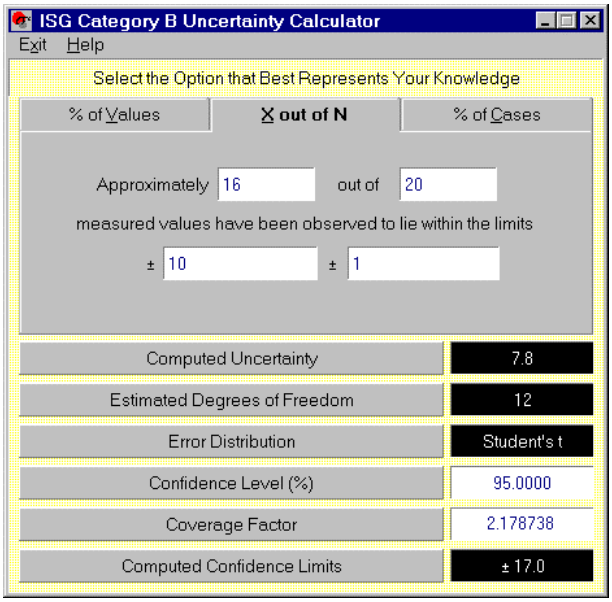

Figure 2. Degrees of freedom estimate for a case where approximately 16 out of 20 values are observed as being within the limits ± 10. In this case, the approximate nature of the estimate is embodied in the secondary limits ±1 and in the binomial character of the estimate. Given the latter, the binomial uncertainty component is given by $\Delta p = p(1 - p) / n$, where $x = 16$, $n = 20$ and $p = x / n$. The format shown is Format 2.

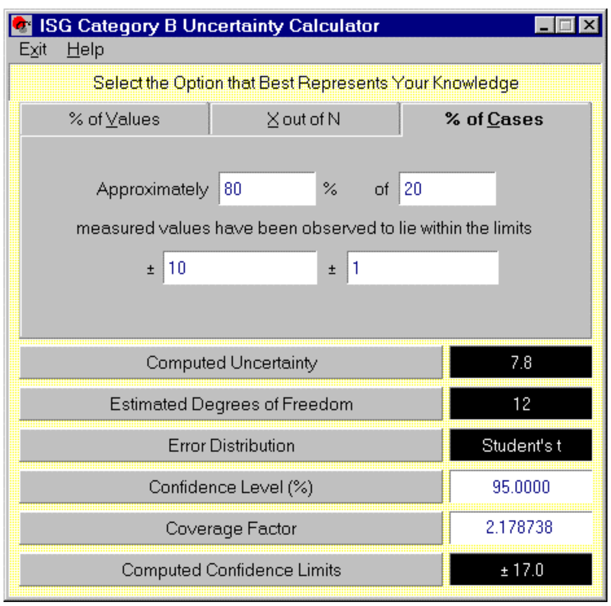

Figure 3. Degrees of freedom estimate for a case where approximately 80% out of 20 values are observed as being within the limits ±10. The approximate nature of the estimate is embodied in the secondary limits ±1 and in the binomial character of the estimate. Given the latter, the binomial uncertainty component is given by $\Delta p = p(1 - p) / n$, where $p = 0.80$ and $n = 20$. The format shown is a variation of Format 2.

COMBINED CATEGORY A AND B UNCERTAINTIES

The procedure for estimating and combining Category A and B estimates is straightforward. We will consider a case where the only errors present are due to random variations in measured value and to the bias of the measuring device.[10] It is assumed that these errors are each normally distributed with zero mean.

The Uncertainty Estimates

As indicated earlier, a Category A uncertainty estimate is equated with the standard deviation of a sample of measurements. Letting x_i represent the *ith* measured value of a sample of n measurements of a variable x, the standard deviation in x is given by

$$s_x = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2} \,, \qquad (21)$$

where the sample mean, $\bar{x}$, is given by

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \,. \qquad (22)$$

In what follows, we will be developing confidence limits for the sample mean value, rather than an isolated measurement of x. Accordingly, we apply the sampling distribution, which makes the Category A uncertainty estimate equal to s_x divided by the square root of n

$$u_A = s_{\bar{x}} = \frac{s_x}{\sqrt{n}} \,. \qquad (23)$$

The variable u_A represents the uncertainty due to random variations that occur during measurement. These random variations will be centered around some unknown "true value" μ plus whatever systematic error is present at the time the sample of measurements is taken. In this example, we assume that this systematic error is due to the bias of the measuring instrument. We estimate the uncertainty in this bias using Eqs. (7) and (8)

$$u_B = \frac{A}{\varphi(p)} \,, \qquad (24)$$

where the limits $\pm A$ are the measuring parameter tolerance limits, p is the measuring parameter in-tolerance probability, and

$$\varphi(p) = \Phi^{-1}\left[(1+p)/2\right] \,,$$

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[10] This is, of course, a simplification. In an actual analysis, other errors, such as resolution error, operator bias, etc. would also have to be accounted for.

as before.

The uncertainty due to the combined random and systematic error, $\varepsilon_A + \varepsilon_B$, is the square root of the variance of the total error

$$u = \sqrt{\mathrm{var}(\varepsilon_A + \varepsilon_B)}$$
$$= \sqrt{u_A^2 + u_B^2 + 2\rho_{AB} u_A u_B} \ ,$$

where ρ_{AB} is the correlation coefficient between ε_A and ε_B. In typical measurements, these errors are statistically independent, and $\rho_{AB} = 0$. Accordingly, we have

$$u = \sqrt{u_A^2 + u_B^2} \ . \qquad (25)$$

The Degrees of Freedom

The degrees of freedom for the estimate u_A is simply

$$\nu_A = n - 1 \, ,$$

while the degrees of freedom for u_B is computed using Eq. (17) or (20)

$$\nu_B \cong \frac{3\varphi^2 A^2}{2\varphi^2 (\Delta A)^2 + \pi A^2 e^{\varphi^2} (\Delta p)^2}$$

or

$$\nu_B \cong \frac{3\varphi^2 A^2}{2\varphi^2 (\Delta A)^2 + 3\pi A^2 e^{\varphi^2} p(1-p)/n} \ .$$

Once ν_A and ν_B are determined, the degrees of freedom for u can be expressed using the Welch-Satterthwaite formula

$$\nu = \frac{u^4}{\sum_{i=1}^{k} u_i^4 / \nu_i} = \frac{u^4}{u_A^4 / \nu_A + u_B^4 / \nu_B} \ .$$

The Confidence Limits

Having computed u and ν, we can now establish confidence limits for the mean value given in Eq. (22). This is done using a t-statistic, designated $t_{\alpha,\nu}$, where α represents a desired confidence level.[11] For two-sided limits with confidence level of 95%, for instance, $\alpha = 0.025$.

The confidence limits serve as upper and lower limits that contain the true value μ (estimated by the mean value $\bar{x}$), with a probability equal to the confidence level, i.e., we say that the interval

$$\bar{x} - t_{\alpha,\nu} u \ \le \ \mu \ \le \ \bar{x} + t_{\alpha,\nu} u$$

contains the true value with $(1 - 2\alpha) \times 100\%$ probability.

[11] Tabulated values of $t_{\alpha,\nu}$ can be found in statistics texts and mathematics handbooks.

CONCLUSION

By obtaining values for the degrees of freedom for Category B uncertainty estimates, we place these estimates on a statistical footing. It is through the medium of the degrees of freedom that the approximate nature of Category B estimates is quantitatively accounted for. Once this has been achieved, Category B estimates can take their place alongside Category A estimates in developing confidence limits and in other activities where the uncertainty estimate is taken to be a standard deviation for an underlying error distribution. This is particularly evident in combining Category A and B estimates into a total uncertainty. Given rigorous values for the degrees of freedom for both Category A and B components, the degrees of freedom for the combined total can be determined using the Welch-Satterthwaite formula. This means that the combined total may also be treated statistically.

A happy consequence of this is that we can rid ourselves of the embarrassment of arbitrary coverage factors that often bear no relationship to confidence levels or anything else of use. In addition, we no longer need to obfuscate the communication of uncertainty analysis results with the term "expanded uncertainty" to mask our inability to handle Category B estimates in a statistical way. Instead, we can return to the use of confidence limits based on considerations of uncertainty and probability.

The foregoing is not meant to imply that the problem of estimating Category B degrees of freedom has been solved and put to bed in this paper. More research is needed in the area of extracting objective data from subjective recollections and in quantifying the lack of knowledge accompanying such data. With regard to the latter, work is required to generalize the methodology presented herein to non-normal distributions (such as may pertain to asymmetric error limits) and to the problem of combining distributions of mixed character.

REFERENCES

[1] *Guide to the Expression of Uncertainty in Measurement*, 3rd Draft, ISO/TAG4/WG3, February 1990.

[2] Taylor, B. and Kuyatt, C., "Guidelines for Evaluating and Expressing the Uncertainty of NIST Measurement Results," NIST Technical Note 1297, U.S. Dept. of Commerce, January 1993.

[3] Proschan, F., "Confidence and Tolerance Intervals for the Normal Distribution," *Proc. Annual American Statistical Association*, Boston, December 1951.

[4] Youden, W., "Uncertainties in Calibration," *Proc. 1962 International Conference on Precision Electromagnetic Measurements*, August 1962.

[5] Eisenhart, C., "Realistic Evaluation of the Precision and Accuracy of Instrument Calibration Systems," J. Res. NBS, Vol. 67C, No. 2, April-June 1963.

[6] Ku, H., "Statistical Concepts in Metrology," *Handbook of Industrial Metrology*, American Society of Tool and Manufacturing Engineers, Prentice-Hall, Inc., New York, 1967.

[7] Ku, H., "Expressions of Imprecision, Systematic Error, and Uncertainty Associated with a Reported Value," *Precision Measurements and Calibration, Statistical Concepts and Procedures*, NBS Special Pub. 300, Vol. 1, Ed. by H. Ku, February 1969.

[8] *UncertaintyAnalyzer*, © 1994-1997 Integrated Sciences Group.

[9] *Category B Degrees of Freedom Estimator*, © 1999 Integrated Sciences Group, Freeware available from http://www.isgmax.com.

[10] Castrup, H., Uncertainty Analysis for Risk Management, *Proc. Measurement Science Conference*, Anaheim, January 1995.

[11] Castrup, H., "Analyzing Uncertainty for Risk Management," *Proc. 49th ASQC Annual Quality Congress*, Cincinnati, May 1995.

[12] Castrup, H., "Uncertainty Analysis and Parameter Tolerancing," *Proc. NCSL Workshop & Symposium*, Dallas, July 1995.