Estimating Parameter Bias Uncertainty

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Uncertainty Techniques and Methods

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Estimating Parameter Bias Uncertainty

- Key Concepts
- Type A Estimation
- Type B Estimation
- Type B Degrees of Freedom
- Type A Estimation
- Bayesian Estimation

Key Concepts

- Measurements are Accompanied by **Measurement Error**
- The Fundamental Measurement Model

$$x_{meas} = x_{true} + \varepsilon_{meas}$$

- The Measurement Error

$$\varepsilon_{meas} = \varepsilon_{bias} + \varepsilon_{random} + \varepsilon_{operator} + \cdots$$

Type A Estimation

- Example:
  - Uncertainty Due to Random Error:

$$u_{random} = \sqrt{\frac{1}{\nu}\sum_{i=1}^{n}(x_i - \overline{x})^2}$$

  - Where

$$x_i = ith \text{ sampled value}$$

$$\overline{x} = \text{sample mean}$$

$$n = \text{sample size}$$

$$\nu = n - 1$$

Key Concepts

Type A Estimation (cont.)

- Confidence Limits $\pm L$ for a Confidence Level of $1 - \alpha$

$$L = t_{\alpha/2,\nu} u_{random}$$

- $t_{\alpha/2,\nu}$ = t-statistic for a significance level of $\alpha/2$ and degrees of freedom ν

Key Concepts

Type B Estimation

- Estimation Procedure
- Containment Limits and Probability
- Uncertainty Definition
- Bias Distributions

See "Estimating and Combining Uncertainties" and "A Comprehensive Comparison of Uncertainty Analysis Tools" at www.isgmax.com/articles_papers.htm .

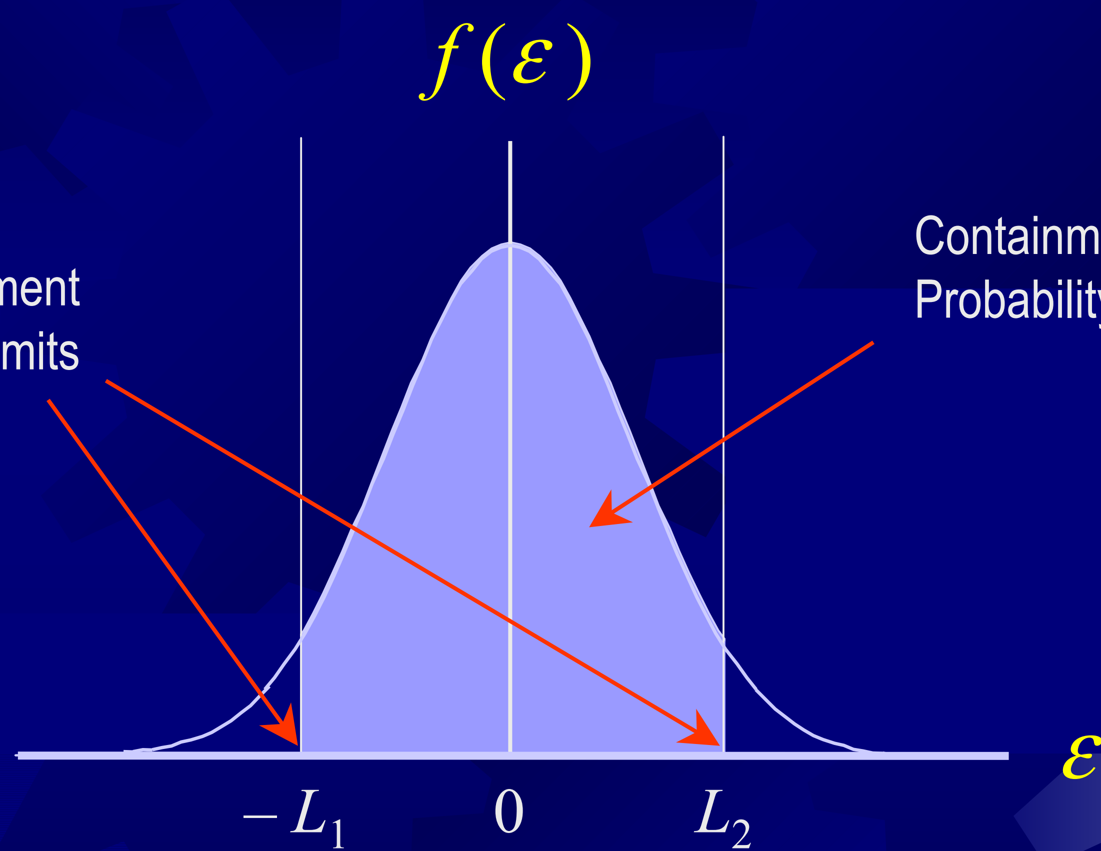
Type B Estimation

Estimation Procedure

- “Reverse” the Type A Estimation Procedure
  - Estimate or Develop Containment Limits
    - Parameter Tolerance Limits
    - Other
  - Estimate a Containment Probability
    - Percent In-tolerance
    - Confidence Level

Type B Estimation

Containment Limits and Probability



The Parameter Bias Distribution

Type B Estimation

Uncertainty Definition

- The standard uncertainty in parameter bias is the standard deviation of the bias distribution.

$$\mathrm{var}(\varepsilon) = \int_{-\infty}^{\infty} f(\varepsilon)\varepsilon^2 d\varepsilon$$

$$u_\varepsilon = \sqrt{\mathrm{var}(\varepsilon)}$$
$$= \sigma_\varepsilon$$

- Apply the appropriate distribution.
- Estimate σ_ε from L_1, L_2 and p.

Type B Estimation

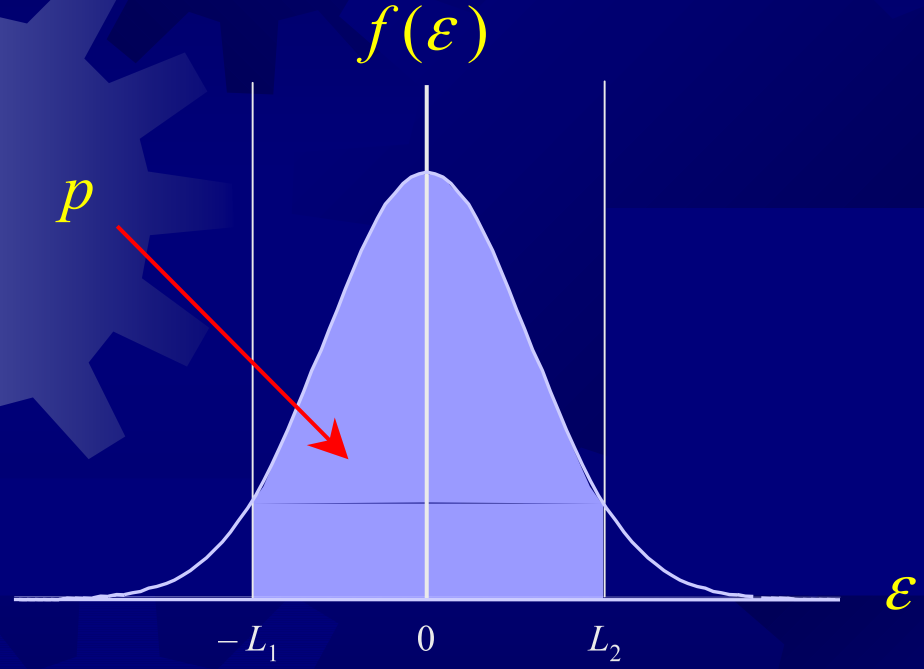
Bias Distributions

- Normal
- Lognormal
- Uniform (Rectangular)
- Triangular
- Trapezoidal
- Quadratic
- Cosine
- Utility
- U-Shaped

See "Distributions for Uncertainty Analysis" at www.isgmax.com/articles_papers.htm.

Bias Distributions

Normal Distribution



the pdf

$$f(\varepsilon) = \frac{1}{\sqrt{2\pi}u_\varepsilon} e^{-\varepsilon^2/2u_\varepsilon^2}$$

solve for u_ε from

$$p = \Phi\left(\frac{L_1}{u_\varepsilon}\right) + \Phi\left(\frac{L_2}{u_\varepsilon}\right) - 1$$

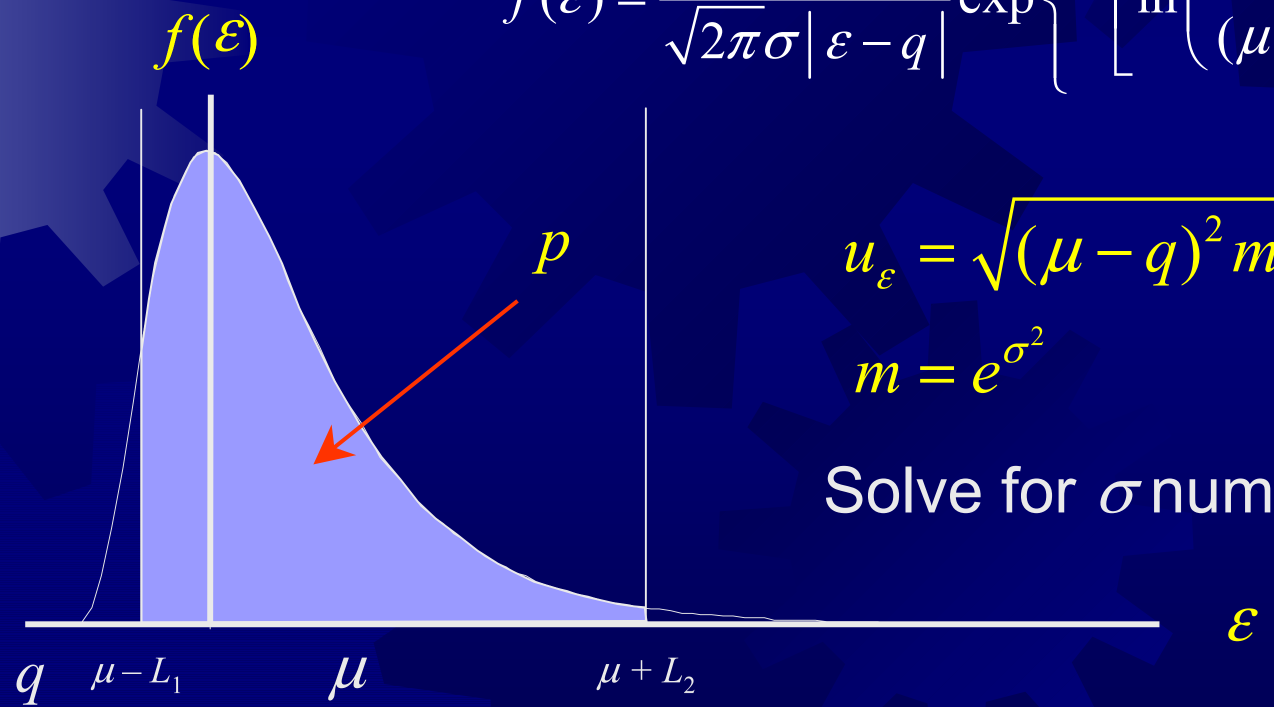
if $L_1 = L_2$ then

$$u_\varepsilon = \frac{L}{\Phi^{-1}\left(\frac{1+p}{2}\right)}$$

Lognormal Distribution

- Useful for Parameters with Asymmetric Tolerances

$$f(\varepsilon)=\frac{1}{\sqrt{2\pi}\sigma\left|\varepsilon-q\right|}\exp\left\{-\left[\ln\left(\frac{\varepsilon-q}{(\mu-q)m}\right)\right]^2\Big/2\sigma^2\right\}$$



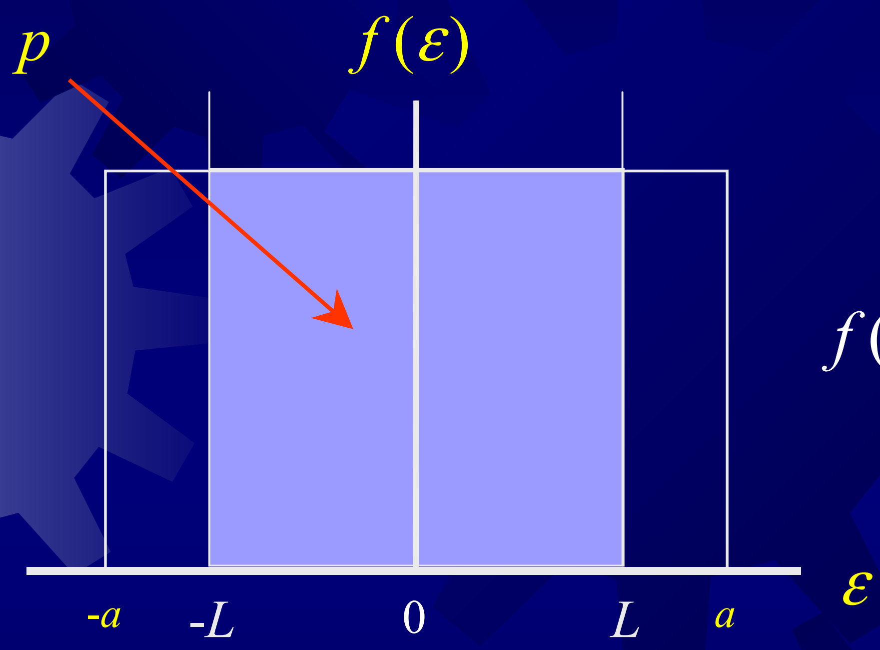
$$u_\varepsilon=\sqrt{(\mu-q)^2m^3(m-1)}$$

$$m=e^{\sigma^2}$$

Solve for σ numerically

Bias Distributions

Uniform (Rectangular) Distribution



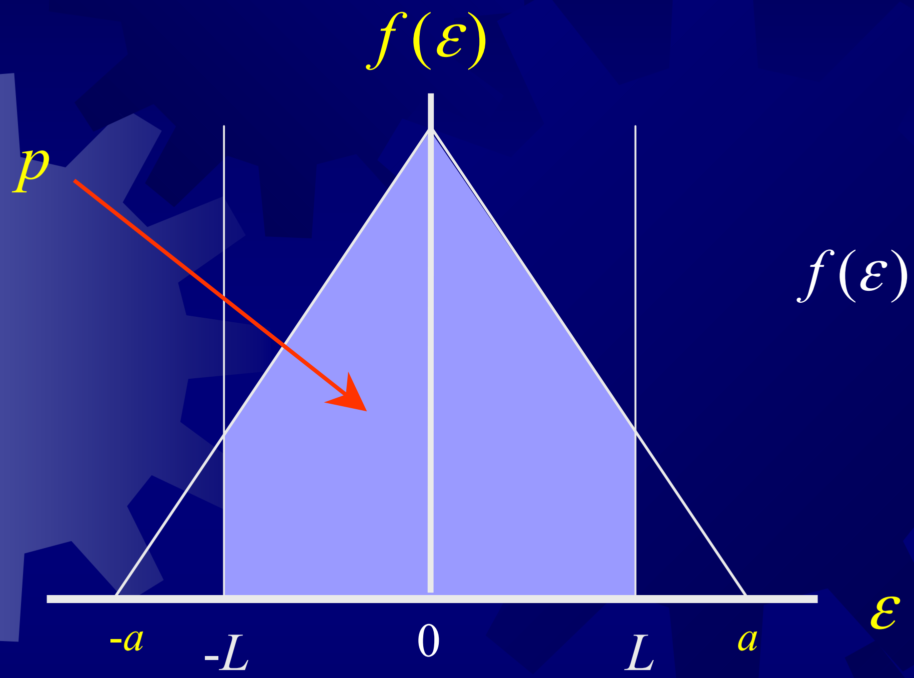
$$f(\varepsilon) = \begin{cases} \frac{1}{2a}, & -a \le \varepsilon \le a \\ 0, & \text{otherwise} \end{cases},$$

$$u_\varepsilon = \frac{a}{\sqrt{3}} \qquad a = \frac{L}{p}, \quad L \le a$$

Not recommended for estimating bias uncertainty

Bias Distributions

Triangular Distribution



$$f(\varepsilon) = \begin{cases} (\varepsilon + a)/a^2, & -a \le \varepsilon \le 0 \\ (a - \varepsilon)/a^2, & 0 \le \varepsilon \le a \\ 0, & \text{otherwise.} \end{cases}$$

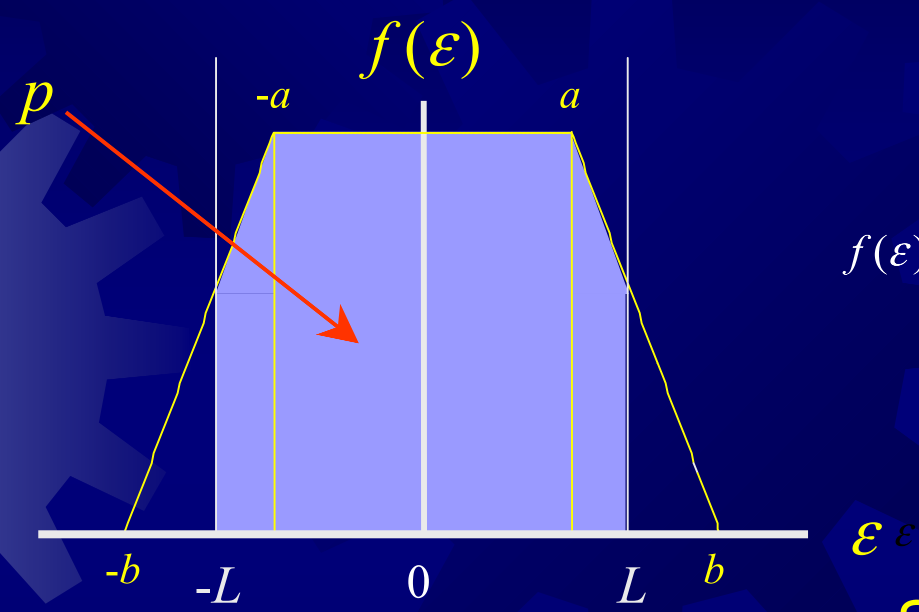
$$u = \frac{a}{\sqrt{6}}$$

$$a = \frac{L}{1 - \sqrt{1 - p}}, \quad L \le a$$

Not recommended for estimating bias uncertainty

Bias Distributions

Trapezoidal Distribution



$$f(\varepsilon)=\begin{cases}\dfrac{\varepsilon+b}{(a+b)(b-a)}, & -b\le\varepsilon\le -a\\ \dfrac{1}{a+b}, & -a\le\varepsilon\le a\\ \dfrac{b-\varepsilon}{(a+b)(b-a)}, & a\le\varepsilon\le b\\ 0\,. & \text{otherwise}\end{cases}$$

Solve for a from

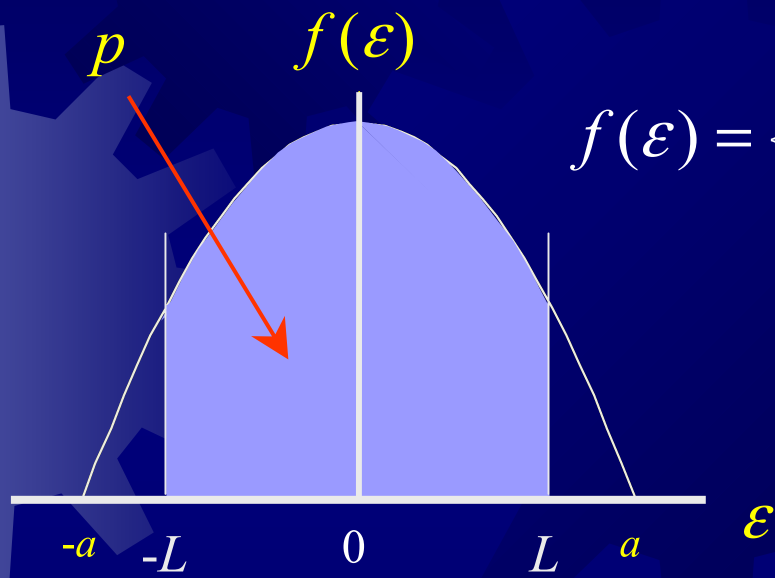
$$u_\varepsilon=\sqrt{\frac{a^2+b^2}{3}}$$

$$\frac{2}{a+b}\left[a+\frac{L-a}{b-a}\left(b-\frac{a+L}{2}\right)\right]-p=0$$

Not recommended for estimating bias uncertainty

Bias Distributions

Quadratic Distribution



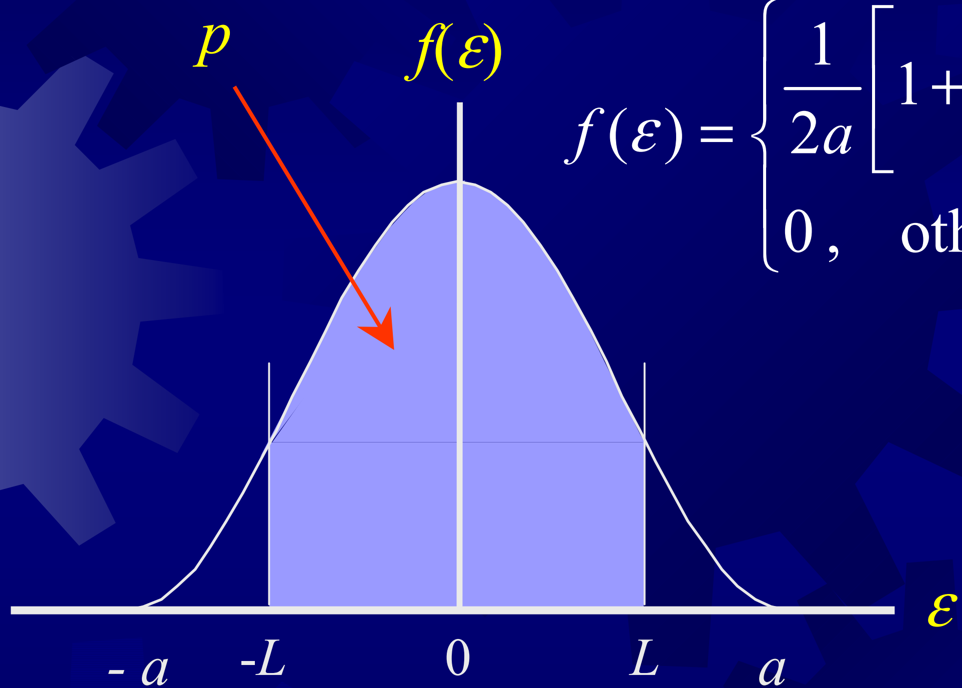
$$f(\varepsilon)=\begin{cases}\dfrac{3}{4a}\left[1-(\varepsilon/a)^2\right], & -a\le\varepsilon\le a\\ 0, & \text{otherwise}\end{cases}$$

Solve for a from

$$u=\frac{a}{\sqrt{5}},\qquad x^3-3x+2p=0,\quad x=L/a\le 1$$

Bias Distributions

Cosine Distribution



$$f(\varepsilon) = \begin{cases} \frac{1}{2a}\left[1+\cos\left(\frac{\pi\varepsilon}{a}\right)\right], & -a \le \varepsilon \le a \\ 0, & \text{otherwise} \end{cases}$$

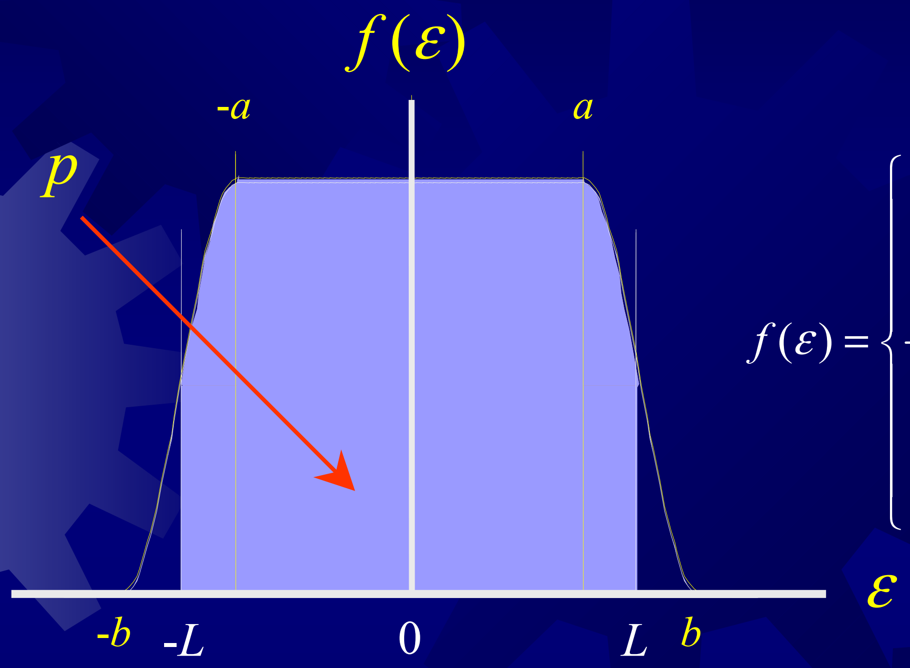
$$u = \frac{a}{\sqrt{3}}\sqrt{1-\frac{6}{\pi^2}}$$

Solve for a from

$$\frac{a}{\pi}\sin(\pi L / a) - ap + L = 0, \quad L \le a$$

Bias Distributions

Utility Distribution



$$f(\varepsilon) = \begin{cases} \dfrac{1}{a+b}, & |\varepsilon| \le a \\ \dfrac{1}{a+b}\cos^2\left[\dfrac{\pi(|\varepsilon|-a)}{2(b-a)}\right], & a \le |\varepsilon| \le b \\ 0, & |\varepsilon| \ge b, \end{cases}$$

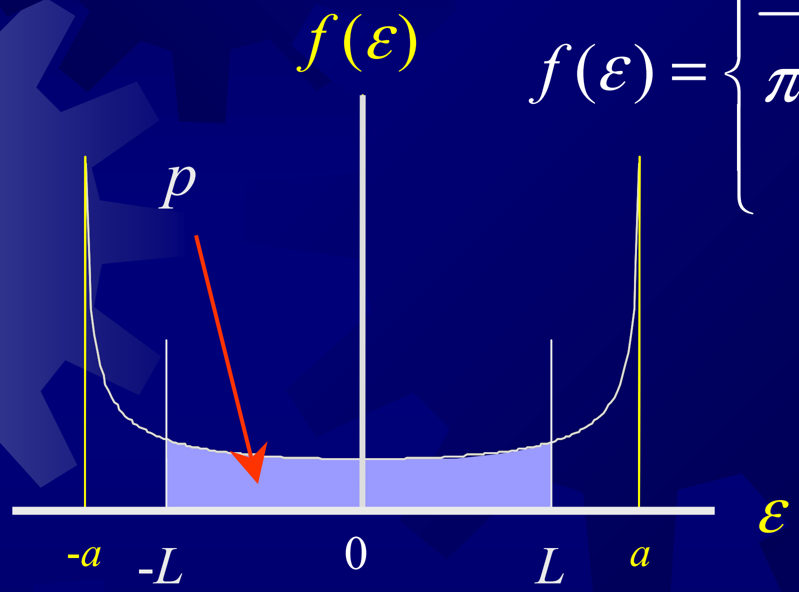
$$u^2 = \frac{1}{3}\frac{a^3+b^3}{a+b} - \frac{2}{\pi^2}(b-a)^2$$

Solve for a from

$$\frac{1}{a+b}\left\{L + \frac{b-a}{\pi}\sin\left[\frac{\pi(L-a)}{b-a}\right]\right\} - p = 0$$

Bias Distributions

U-Shaped Distribution



$$f(\varepsilon) = \begin{cases} \dfrac{1}{\pi\sqrt{a^2 - \varepsilon^2}}, & -a < \varepsilon < a \\ 0, & \text{otherwise} \end{cases},$$

$$u = \frac{a}{\sqrt{2}}$$

Solve for a from

$$a = \frac{L}{\sin(\pi p / 2)}, \quad L \le a$$

Estimating Parameter Bias Uncertainty

Type B Degrees of Freedom

- What is it?
  - Type A Estimate: $n - 1$
  - The larger the sample size, the larger the degrees of freedom
  - The larger the sample size, the more information we have for making the uncertainty estimate

The degrees of freedom is a number that quantifies the amount of "knowledge" on which an uncertainty estimate is based.

Statistical Degrees of Freedom

- We have available a rigorous method for obtaining the degrees of freedom for Type B estimates.
- We can speak of population standard deviations and degrees of freedom for Type B estimates and can compute confidence limits.
- Type B estimates and degrees of freedom can be obtained using **Type B Analysis Formats**.

See "Note on the Type B Degrees of Freedom Equation" at www.isgmax.com/articles_papers.htm.

Type B Degrees of Freedom

Analysis Formats

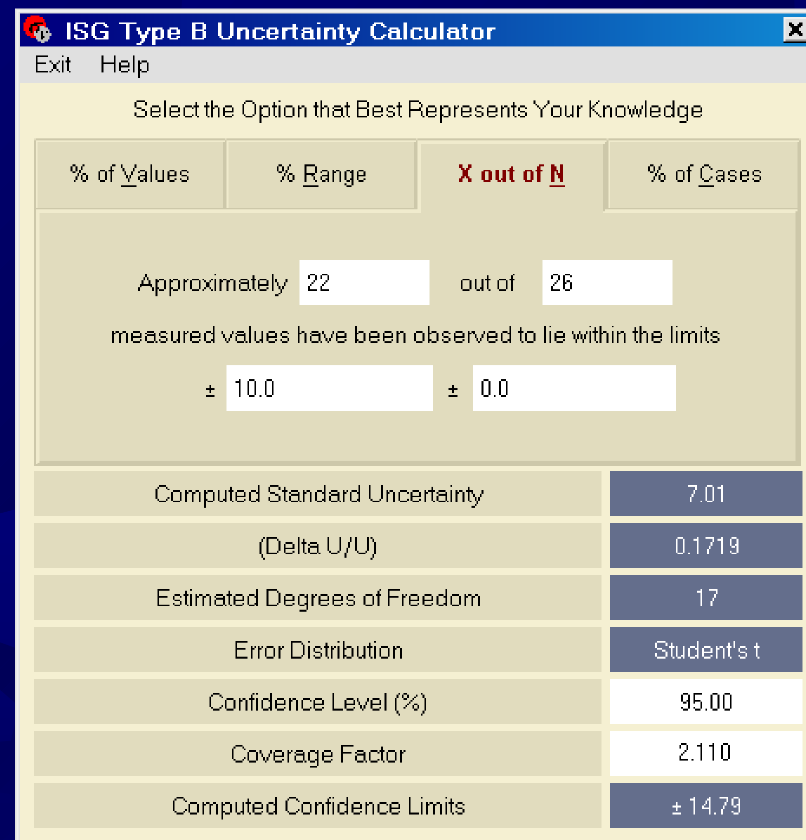
Format 1: Approximately $C\%$ *of values* $(\pm \Delta c\%)$ have been observed to lie within $\pm L\ (\pm \Delta L)$

Format 2: Approximately n out of N values have been observed to lie within $\pm L\ (\pm \Delta L)$

Format 3: Approximately $C\%$ of N values have been observed to lie within $\pm L\ (\pm \Delta L)$

Type B Uncertainty Calculator

- Freeware Application
- Available at www.isgmax.com
- Methodology Available from Type B Uncertainty Calculator Help



Type A Estimation

Analysis of Variance

- Estimate the Standard Deviation of the Bias of a Given Parameter of a Population of UUT Items
  - UUTs Calibrated by a Sample of Operators at End-of-Period
    - Establishes a common uncertainty growth base
  - Each Operator Calibrates Each UUT at Least Once
    - If an operator calibrates a UUT more than once, combine his or her sampled values into one sample.

Type A Estimation

The ANOVA Model

Imagine that m operators take samples of measurements of a particular UUT parameter. Suppose that the samples are taken on n independent UUT items, using a common measurement reference.

The basic two-factor analysis of variance (ANOVA) model is

$$y_{ijk} = \mu + \alpha_i + \beta_j + \delta_{ij} + \varepsilon_{ijk}$$

$y_{ijk} =$ the kth measurement of the value of the parameter of the ith UUT taken by the jth operator

$n_{ij} =$ the sample size for the measurements made by the jth operator on the ith UUT

Type A Estimation

The ANOVA Model (cont.)

- Variables of the Model
  - μ = expected mean value of measurements of the UUT parameter
  - ε_{ijk} = random error in the kth measurement by the jth operator on the parameter of the ith UUT item
  - Remaining variables to be defined presently

Type A Estimation

The ANOVA Model (cont.)

- Computed Variables:
  - Overall Mean

$$y = \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{1}{n_{ij}}\sum_{k=1}^{n_{ij}} y_{ijk} = \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} y_{ij}$$

  - Mean of the sample taken by the jth operator on the ith UUT

$$y_{ij} = \frac{1}{n_{ij}}\sum_{k=1}^{n_{ij}} y_{ijk}$$

  - Mean of the measurements taken on the ith UUT

$$y_i = \frac{1}{n}\sum_{i=1}^{n} y_{ij}$$

  - Mean of the measurements taken by the jth operator

$$y_j = \frac{1}{m}\sum_{j=1}^{m} y_{ij}$$

Type A Estimation

The ANOVA Model (cont.)

- Estimates of the Variables of the Model

$$y_{ijk} = \mu + \alpha_i + \beta_j + \delta_{ij} + \varepsilon_{ijk}$$

$$= y_{ij} + \varepsilon_{ijk} = y_{ij} + (y_{ijk} - y_{ij})$$

$$\hat{\mu} = y$$

$$\hat{\alpha}_i = y_i - y$$

$$\hat{\beta}_j = y_j - y$$

$$\hat{\delta}_{ij} = y_{ij} - y_i - y_j + y$$

$$\hat{\varepsilon}_{ijk} = y_{ijk} - y_{ij}$$

Type A Estimation

The ANOVA Model (cont.)

- ANOVA Sums of Squares

$$SS_A = \sum_{i=1}^{n} n_i \hat{\alpha}_i^2$$

$$SS_B = \sum_{j=1}^{m} n_j \hat{\beta}_j^2$$

$$SS_{AB} = \sum_{i=1}^{n} \sum_{j=1}^{m} n_{ij} \hat{\delta}_{ij}^{\ 2}$$

$$SS_E = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{n_{ij}} \hat{\varepsilon}_{ijk}^{\ 2}$$

where

$$n_i = \sum_{j=1}^{m} n_{ij}$$

$$n_j = \sum_{i=1}^{n} n_{ij}$$

Type A Estimation

The ANOVA Model (cont.)

- Base Variances

$$MS_A = \frac{SS_A}{n-1}$$

$$MS_B = \frac{SS_B}{m-1}$$

$$MS_{AB} = \frac{SS_{AB}}{(n-1)(m-1)}$$

$$MS_E = \frac{SS_E}{\sum_{i=1}^{n}\sum_{j=1}^{m}(n_{ij}-1)}$$

The ANOVA Model

Analysis Results

Variable	Description	Estimate
R	*Repeatability Uncertainty*	$\sqrt{MS_E}$
AV	*Operator Bias Uncertainty*	$\sqrt{m(MS_B - MS_{AB}) \Big/ \sum_{j=1}^{m} n_j}$
PV	*UUT Bias Uncertainty*	$\sqrt{n(MS_A - MS_{AB}) \Big/ \sum_{i=1}^{n} n_i}$
I	*Operator-Part Interaction*	$\sqrt{nm(MS_{AB} - MS_E) \Big/ \sum_{i=1}^{n}\sum_{j=1}^{m} n_{ij}}$
$R\&R$	*Process Uncertainty*	$\sqrt{R^2 + (PV)^2 + (AV)^2 + (I)^2}$

The ANOVA Model

Measurement Reference Bias Uncertainty

- Use the Same Expressions as in Estimating UUT Parameter Bias Uncertainty
- The Index j ranges over m independent MTE Items instead of m independent UUT Items
- The Variable PV (Part Variation) Becomes EV (Equipment Variation)

Estimating Parameter Bias Uncertainty

Bayesian Estimation

- Assemble *a priori* knowledge
- Obtain Measurement Results
- Develop *a posteriori* knowledge
  - Estimate Parameter Biases
  - Estimate Bias Uncertainties
  - Estimate Parameter In-Tolerance Probabilities

See "Analytical Metrology SPC Methods for ATE Implementation" at www.isgmax.com/articles_papers.htm.

Bayesian Estimation

State of Knowledge

- *a priori* Knowledge
  - Standard Deviation for UUT Parameter Bias
  - Standard Deviation for Measurement Reference Bias
- Measurement Results
  - Single Value or Mean Value of Measurements
    - Made by Measurement Reference
    - Made by UUT Parameter
    - Made by Both
- *a posteriori* Knowledge
  - UUT Bias Estimate
  - UUT Bias Uncertainty
  - UUT In-Tolerance Probability
  - Same for Measurement Reference

Bayesian Estimation

a priori Knowledge

- Usually Developed from Type B Estimates
- Standard Deviation for the UUT Parameter Bias Distribution

$$u_{UUT} = \frac{L_{UUT}}{\Phi^{-1}\left(\frac{1 + R_{UUT}}{2}\right)}$$

- Standard Deviation for the Measurement Reference Bias Distribution

$$u_{ref} = \frac{L_{ref}}{\Phi^{-1}\left(\frac{1 + R_{ref}}{2}\right)}$$

Bayesian Estimation

Measurement Results

$x =$ UUT Measurement(s) or Nominal Value

$$u_x = u_{UUT}$$

$y =$ Reference Standard Measurement or Stated Value

$$u_y = \sqrt{u_{ref}^2 + \frac{s_m^2}{n_m} + u_{process}^2}$$

$z =$ “Bayesian” Random Variable

$$z = x - y$$

Bayesian Estimation

a posteriori Knowldge

- UUT Bias Estimate (β) $\quad \beta = \frac{r^2}{1+r^2} z \quad r = u_x / u_y$

- UUT Bias Uncertainty (u_x / α) $\quad \alpha = \sqrt{1+r^2}$

- UUT In-Tolerance Probability

$$P = \frac{1}{\sqrt{2\pi}\left(u_x / \alpha\right)} \int_{-L_{UUT}}^{L_{UUT}} e^{-(\varepsilon-\beta)^2 / 2\left(u_x / \alpha\right)^2} d\varepsilon$$

$$= \Phi\left(\frac{L_{UUT} + \beta}{u_x / \alpha}\right) + \Phi\left(\frac{L_{UUT} - \beta}{u_x / \alpha}\right) - 1$$

Recap

- Type B Estimate
  - E.g., Normal Distribution:

$$u_b = \frac{L}{\Phi^{-1}\left(\frac{1+p}{2}\right)}$$

- ANOVA Estimate:

$$u_b = \sqrt{n(MS_A - MS_{AB}) \Big/ \sum_{i=1}^{n}\sum_{j=1}^{m} n_{ij}}$$

- Bayesian Estimate:

$$u_b = u_x / \alpha \qquad u_x = u_{UUT} \qquad \alpha = \sqrt{1 + r^2} \qquad r = u_x / u_y$$

Documentation

- A written paper will be available at a later date at

www.isgmax.com