Uncertainty Analysis for Alternative Calibration Scenarios[1]

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Abstract

Calibrations are performed to obtain an estimate of the value or bias of selected unit-under-test (UUT) attributes. In general, calibrations are not considered complete without statements of the uncertainty in these estimates. Developing these statements requires accounting for all relevant sources of measurement error and assembling these errors in a way that yields viable uncertainty estimates. Frequently, there is confusion regarding which error sources to include and how to assemble them. Much of this confusion can be eliminated by both a rigorous examination of the objective of each UUT attribute calibration and a consideration of the corresponding measurement configuration or "scenario."

In this paper, the calibration of a UUT attribute[2] is examined within the context of four scenarios. Each scenario yields a calibration result and a description of measurement process errors that accompany this result. This information is summarized and then employed to obtain an uncertainty estimate in the calibration result. The approach taken is one in which the uncertainty estimate can be applied to estimate measurement decision risk, UUT attribute bias and in-tolerance probability. Examples are given to illustrate concepts and procedures.

Introduction

This paper discusses information obtained from measurements made during calibration and the application of this information to measurement decision risk analysis in the context of four calibration scenarios:

1. The measurement reference (MTE) measures the value of a passive attribute of the unit under test (UUT).
2. The UUT measures the value of a passive reference attribute of the MTE.

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[2] An attribute is a measurable characteristic, feature or aspect of an object or substance.

3. The UUT and MTE each provide an "output" or "stimulus" for comparison using a comparator.
4. The UUT and MTE both measure the value of an attribute of a common artifact that provides an output or stimulus.

The information obtained includes an observed value, referred to as a "measurement result" or "calibration result," and an estimated uncertainty in the measurement error. For each scenario, a measurement equation is given that is applicable to the manner in which calibrations are performed and calibration results are recorded or interpreted.

The measurement scenarios turn out to be simple and intuitive. In each, the measurement result and the measurement error are separable, allowing the estimation of measurement uncertainty. For the purposes of this paper, it is assumed in each scenario that the measurement result is an estimate of the value of the bias of the UUT attribute.

Basic Notation

The subscripts and variables designators in this paper are summarized in Table 1.

Table 1. Basic Notation

Notation	Description
e	an individual measurement process error, such as repeatability, resolution error, etc.
ε	combined errors comprised of individual measurement process errors
m	measurement
b	bias
cal	calibration
$true$	true value
n	nominal value

With this notation, for example, measurement error is represented by the quantity ε_m, the error in a calibration result by ε_{cal} and the bias in the UUT attribute is represented by the quantity $e_{UUT,b}$.

As stated in the introduction, specific measurement equations will be given for each calibration scenario. In each equation, quantities relating to the UUT are indicated with the notation x and quantities relating to the MTE with the notation y. For example, in Scenario 1, where the MTE directly measures the value of the UUT attribute, the relevant measurement equation is

$$y = x_{true} + \varepsilon_m , \tag{1}$$

where y represents a measurement taken with the MTE, x_{true} is the true value of the UUT attribute and ε_m is the measurement error. Variations of Eq. (1) will be encountered throughout this paper.

Measurement Uncertainties

Measurement errors and parameter biases are random variables that follow statistical distributions. Each distribution is a relationship between the value of an error and its probability of occurrence. Distributions for errors that are tightly constrained correspond to low uncertainty, while distributions for errors that are widely spread correspond to high uncertainties. Mathematically, the uncertainty due to a particular error is equated to the spread in its distribution. This spread is just the distribution standard deviation which is defined as the square root of the distribution variance [1-3]. Letting u represent uncertainty and "var(ε)" the statistical variance of the distribution of an error ε, we write

$$u = \sqrt{\mathrm{var}(\varepsilon)} \; . \tag{2}$$

This expression will be used in this paper as a template for estimating measurement process uncertainties encountered in the various calibration scenarios.

Measurement Error Sources

Typically, calibration scenarios feature the following set of measurement process errors or "error sources."

$e_{MTE,b}$ = bias in the measurement reference

e_{rep} = repeatability or "random" error

e_{res} = resolution error

e_{op} = operator bias

e_{other} = other measurement error, such as that due to environmental corrections, ancillary equipment variations, response to adjustments, etc.

Measurement Reference Bias

The error in a measurement reference attribute, at any instant in time, is composed of a systematic component and a random component. The systematic component is called "attribute bias." Attribute bias is an error component that persists from measurement to measurement during a "measurement session." Attribute bias excludes resolution error, random error, operator bias and other sources of error that are not properties of the attribute.[3]

Repeatability

Repeatability is a random error that manifests itself as differences in measured value from measurement to measurement during a measurement session. It should be said that random variations in UUT attribute value and random variations due to other causes are not separable from random variations in the value of the MTE reference attribute or any other error source. Consequently, whether e_{rep} manifests itself in a sample of measurements made by the MTE or by the UUT, it must be taken to represent a "measurement process error" rather than an error attributable to any specific influence.

[3] For purposes of discussion, a measurement session is considered to be an activity in which a measurement or sample of measurements is taken under fixed conditions, usually for a period of time measured in seconds, minutes or, at most, hours.

Resolution Error

Reference attributes and/or UUT attributes may provide indications of sensed or stimulated values with some finite precision.

For example, a voltmeter may indicate values to four, five, six, etc., significant digits. A tape measure may provide length indications in meters, centimeters and millimeters. A scale may indicate weight in terms of kg, g, mg, etc. The smallest discernible value indicated in a measurement comprises the resolution of the measurement.

The basic error model for resolution error is

$$x_{indicated} = x_{sensed} + e_{res},$$

where x_{sensed} is a "measured" value detected by a sensor or provided by a stimulus, $x_{indicated}$ is the indicated representation of x_{sensed} and e_{res} is the resolution error.

Operator Bias

Because of the potential for operators to acquire measurement information from an individual perspective or to produce a systematic bias in a measurement result, it sometimes happens that two operators observing the same measurement result will systematically perceive or produce different measured values. The systematic error in measurement due to the operator's perspective or other tendency is referred to as **Operator Bias**.

Operator bias is a "quasi-systematic" error, the error source being the perception of a human operator. While variations in human behavior and response lend this error source a somewhat random character, there may be tendencies and predilections inherent in a given operator that persist from measurement to measurement.

The random contribution is included in the random error source discussed earlier. The systematic contribution is the operator bias.

Repeatability and Resolution Error

It is sometimes argued that repeatability is a manifestation of resolution error. To address this point, imagine three cases. In the first case, values obtained in a random sample of measurements take on just two values and the difference between them is equal to the smallest increment of resolution. If this is the case, we can conclude that "background noise" random variations are occurring that are beyond the resolution of the measurement. If so, we cannot include repeatability as an error source but must acknowledge that the apparent random variations are due to resolution error. Accordingly, the uncertainty due to resolution error should be included in the total measurement uncertainty but the uncertainty due to repeatability should not.

In the second case, values obtained in a random sample are seen to vary in magnitude substantially greater than the smallest increment of resolution. In this case, repeatability cannot be ignored as an error source. In addition, since each sampled value is subject to resolution error, resolution error should also be separately accounted for. Accordingly, the total measurement uncertainty must include contributions from both repeatability uncertainty and resolution uncertainty.

The third case is not so easily dealt with. In this case. values obtained in a random sample of measurements are seen to vary in magnitude somewhat greater than the smallest increment of resolution but not substantially greater. In this case, we perceive an error due to repeatability that is separable from resolution error but is partly due to it. In this case, it becomes a matter of opinion as to whether to include repeatability and resolution error in the total measurement error. Until a clear solution to the problem is found, it is the opinion of the authors that both should be included in this case.

Other Error

Other measurement error is a catch-all label applied to errors such as those due to environmental corrections, ancillary equipment variations, response to adjustments, etc. For example, suppose that "other" error is due to an environmental factor, such as temperature, vibration, humidity or stray emf. In many cases, as in accommodating thermal expansion, the effect of an environmental factor can be corrected for. Such corrections usually rely on a measurement of the driving environmental factor.

When this happens, the parameter that measures the environmental factor is referred to as an ancillary parameter. An example would be a thermometer reading used to correct for thermal expansion in the measurement reference and the UUT attributes. An ancillary parameter is subject to error as is any other parameter, and this error can lead to an error in the environmental correction. The uncertainty in the error of the correction is a function of the uncertainty in the error due to the environmental factor.

For a more complete discussion on uncertainties due to environmental and other ancillary factors, see Ref [1].

Calibration Error and Measurement Error

For the scenarios discussed in this paper, the result of a calibration is taken to be the estimation of the bias $e_{UUT,b}$ of the UUT attribute. The error in the calibration result is represented by the quantity ε_{cal}. In all scenarios, the uncertainty in the estimation of $e_{UUT,b}$ is computed as the uncertainty in ε_{cal}. For some scenarios, ε_{cal} is synonymous with the measurement error ε_m or its negative $-\varepsilon_m$. In other scenarios, as in Scenario 2 where the UUT measures the MTE attribute, ε_m includes $e_{UUT,b}$. Since $e_{UUT,b}$ cannot be included in ε_{cal}, the latter of which is the error in the estimation of $e_{UUT,b}$, we have a situation where ε_{cal} and ε_m may not be of the same sign or magnitude.

UUT Attribute Bias

For calibrations, it is tacitly assumed that the UUT attribute of interest is assigned some design or "nominal" value x_n. The difference between the UUT attribute's true value, x_{true}, and the nominal value x_n is the UUT attribute's bias $e_{UUT,b}$. Accordingly, we can write

$$x_{true} = x_n + e_{UUT,b}. \tag{3}$$

In some cases, the UUT is a passive item, such as a gage block or weight, whose attribute of interest is a simple characteristic like length or mass. In other cases the UUT is an active device, such as a voltmeter or tape measure, whose attribute consists of a reading or other output, like voltage or measured length. In the former case, the concepts of true value and nominal value are straightforward. In the latter case, some comment is needed.

As stated earlier, we consider the result of a calibration to be an estimate the quantity $e_{UUT,b}$. From Eq. (3), we can readily appreciate that, if we can assign the UUT a nominal value x_n, estimating x_{true} is equivalent to estimating $e_{UUT,b}$. Additionally, we acknowledge that $e_{UUT,b}$ is an "inherent" property of the UUT, independent of its resolution, repeatability or other characteristic dependent on its application or usage environment. Accordingly, if the UUT's nominal value consists of a measured reading or other actively displayed output, the UUT bias must be taken to be the difference between the true value of the quantity being measured and the value internally sensed by the UUT, with appropriate environmental or other adjustments applied to correct this value to reference (calibration) conditions.

For example, imagine that the UUT is a steel yardstick whose length is a random variable following a statistical distribution with a standard deviation arising from variations in the manufacturing process. Imagine now that the UUT is used under specified nominal environmental conditions. While under these conditions, repeatability, resolution error, operator bias and other error sources may come into play, the bias of the yardstick is systematically present, regardless of whatever chance relationship may exist between the length of the measured object, the closest observed "tick mark," the temperature of the measuring environment, the perspective of the operator, and so on.

MTE Bias

The value of the reference attribute of the MTE, against which the value of the UUT attribute is compared, has an inherent deviation $e_{MTE,b}$ from its nominal value or a value stated in a calibration certification or other reference document. Letting y_{true} represent the true value of the MTE attribute and letting y_n represent the MTE attribute nominal or assumed value, we have

$$y_{true} = y_n + e_{MTE,b} \,. \tag{4}$$

In some cases, the MTE is a passive item, such as a gage block or weight, whose attribute is a simple characteristic like length or mass. In other cases the MTE is an active device, such as a voltmeter or tape measure, whose attribute consists of a reading or other output, like voltage or measured length. In either case, it is important to bear in mind that $e_{MTE,b}$ is an inherent property of the MTE, exclusive of other errors such as MTE resolution or the repeatability of the measurement process. It may vary with environmental deviations, but can usually be adjusted or corrected to some reference set of conditions. An example of such an adjustment is given in Scenario 1 below.

Calibration Scenarios

The four calibration scenarios identified in this paper's introduction are described in detail in the following discussions. The descriptions are not offered to serve as recipes to be followed as dogma but are instead intended to provide guidelines for developing uncertainty estimates relevant to each scenario. It is hoped that the structure and content of each description will assist in developing whatever mathematical customization is needed for specific measurement situations.

In each scenario, we have a measurement of $e_{UUT,b}$, denoted δ, and a calibration error ε_{cal}. The general expression is

$$\delta = e_{UUT,b} + \varepsilon_{cal} .$$

Since $e_{UUT,b}$ is the quantity being estimated by calibration, as discussed earlier, the uncertainty of interest is understood to be the uncertainty in δ given the UUT bias $e_{UUT,b}$. Then, by Eq. (2), we have

$$\begin{aligned} u_{cal} &= \sqrt{\text{var}(\delta)} \\ &\sqrt{\text{var}(e_{UUT,b} + \varepsilon_{cal})} \\ &= \sqrt{\text{var}(\varepsilon_{cal})} . \end{aligned} \qquad (5)$$

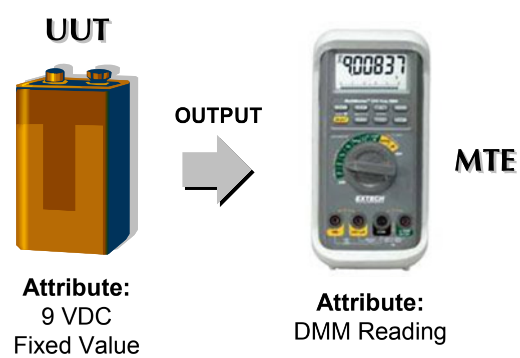

Figure 1. Scenario 1 – The MTE measures the value of a UUT Attribute. The output is the battery voltage.

Scenario 1: The MTE Measures the UUT Attribute Value

In this scenario, the UUT is a passive device whose calibrated attribute provides no reading or other metered output. Its output may consist of a generated value, as in the case of a voltage reference, or a fixed value, as in the case of a gage block.[4] The measurement Eq. is repeated from Eq. (1) as

$$y = x_{true} + \varepsilon_m , \qquad (6)$$

where y is the measurement result obtained with the MTE, x_{true} is the true value of the UUT attribute and ε_m is the measurement error.

The "measured" value provided by the UUT is its nominal value x_n, given in Eq. (3), so that

$$x_{true} = x_n + e_{UUT,b}. \qquad (7)$$

[4] Cases where the MTE measures the value of a metered or other UUT attribute exhibiting a displayed value are covered later as special instances of Scenario 4.

Substituting Eq. (7) in Eq. (6), we write the measurement equation as

$$y = x_n + e_{UUT,b} + \varepsilon_m .$$

The difference $y - x_n$ is a measurement of the UUT bias $e_{UUT,b}$. We denote this quantity by the variable δ and write

$$\begin{aligned} \delta &= y - x_n \\ &= e_{UUT,b} + \varepsilon_m \\ &= e_{UUT,b} + \varepsilon_{cal} . \end{aligned} \tag{8}$$

For this scenario, the calibration error ε_{cal} is equal to the measurement error ε_m and is comprised of MTE bias, measurement process repeatability, MTE resolution error, operator bias, etc. The appropriate expression is

$$\varepsilon_{cal} = e_{MTE,b} + e_{rep} + e_{res} + e_{op} + e_{other} . \tag{9}$$

Since the UUT is a passive device in this scenario, resolution error, and operator bias arise exclusively from the use of the MTE, i.e., $e_{UUT,res}$ and $e_{UUT,op}$ are zero. In addition, the uncertainty due to repeatability is estimated from a random sample of measurements taken with the MTE. Still, variations in UUT attribute value may contribute to this estimate. However, random variations in UUT attribute value and random variations due to other causes are not separable from random variations due to the MTE. Consequently, as stated earlier, e_{rep} must be taken to represent a "measurement process error" rather than an error attributable to any specific influence. Given these considerations, the error sources e_{rep}, e_{res} and e_{op} in Eq. (9) are

$$\begin{aligned} e_{rep} &= e_{MTE,rep} \\ e_{res} &= e_{MTE,res} \\ e_{op} &= e_{MTE,op} , \end{aligned} \tag{10}$$

where $e_{MTE,rep}$ represents the repeatability of the measurement process. The "MTE" part of the subscript indicates that the uncertainty in the error will be estimated from a sample of measurements taken by the MTE.

In some cases, the error source e_{other} may need some additional thought. For example, suppose that e_{other} arises from corrections ensuing from environmental factors, such as thermal expansion. If measurements are made of the length of a UUT gage block using an MTE reference "super mike," it may be desired to correct measured values to those that would be attained at some reference temperature, such as 20 °C.

Let $\delta_{UUT,env}$ and $\delta_{MTE,env}$ represent thermal expansion corrections to the gage block and super mike dimensions, respectively. Then the mean value of the measurement sample would be corrected by an amount equal to[5]

$$\delta_{env} = \delta_{MTE,env} - \delta_{UUT,env} , \tag{11}$$

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[5] The form of this expression arises from the fact that thermal expansion of the gage block results in an inflated gage block length, while thermal expansion of the supermike results in applying additional thimble adjustments to narrow the gap between the anvil and the spindle, resulting in a deflated measurement reading.

and the error in the corrections would be written

$$\begin{aligned} e_{other} &= e_{env} \\ &= e_{MTE,env} - e_{UUT,env} \,. \end{aligned} \tag{12}$$

From Eqs. (8) and (5), we can write the uncertainty in the calibration result δ as

$$u_{cal} = \sqrt{\text{var}(\varepsilon_{cal})} \,, \tag{13}$$

where

$$\begin{aligned} \text{var}(\varepsilon_{cal}) &= \text{var}(e_{MTE,b}) + \text{var}(e_{rep}) + \text{var}(e_{res}) + \text{var}(e_{op}) + \text{var}(e_{other}) \\ &= u_{MTE,b}^2 + u_{rep}^2 + u_{res}^2 + u_{op}^2 + u_{other}^2 \,, \end{aligned} \tag{14}$$

and

$$\begin{aligned} u_{rep} &= u_{MTE,rep} \\ u_{res} &= u_{MTE,res} \\ u_{op} &= u_{MTE,op} \,. \end{aligned} \tag{15}$$

For this scenario, no correlations are present between the error sources shown in Eq. (14). Hence the simple RSS uncertainty combination. This may not be true for correlations *within* some of the terms, as may be the case when $e_{other} = e_{env}$. In this case, we would have

$$u_{other} = \sqrt{u_{MTE,env}^2 + u_{UUT,env}^2 - 2\rho_{env} u_{MTE,env} u_{UUT,env}} \,. \tag{16}$$

If the same temperature measurement device (e.g., thermometer) is used to make both the UUT and MTE corrections, we would have $\rho_{env} = 1$, and

$$\begin{aligned} u_{other} &= \sqrt{u_{MTE,env}^2 + u_{UUT,env}^2 - 2u_{MTE,env} u_{UUT,env}} \\ &= \left| u_{MTE,env} - u_{UUT,env} \right| . \end{aligned} \tag{17}$$

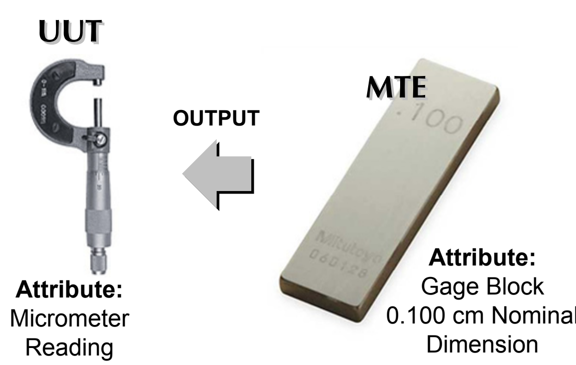

Figure 2. Scenario 2 – The UUT measures the value of an MTE attribute. The output is the gage block dimension.

Scenario 2: The UUT Measures the MTE Attribute Value

In this scenario, the MTE is a passive device whose reference attribute provides no reading or other metered output. Its output may consist of a generated value, as in the case of a voltage reference, or a fixed value, as in the case of a gage block.[6] The measurement equation is a variation of Eq. (1)

$$x = y_{true} + \varepsilon_m \, , \tag{18}$$

where x is the value measured by the UUT, y_{true} is the true value of the MTE attribute being measured and ε_m is the measurement error. Denoting the nominal or indicated value of the MTE as y_n, we can write

$$y_{true} = y_n + e_{MTE,b} \, , \tag{19}$$

where $e_{MTE,b}$ is defined in Eq. (4). Substituting Eq. (19) in Eq. (18) gives

$$x = y_n + e_{MTE,b} + \varepsilon_m \, , \tag{20}$$

and

$$\delta = e_{MTE,b} + \varepsilon_m \tag{21}$$

where δ is the measurement of the UUT bias, given by

$$\delta = x - y_n \, . \tag{22}$$

For this scenario, the measurement error is given by

$$\varepsilon_m = e_{UUT,b} + e_{rep} + e_{res} + e_{op} + e_{other} \, , \tag{23}$$

where $e_{UUT,b}$ is the UUT bias defined in Eq. (3), e_{rep} is the repeatability of the measurement process as evidenced in the sample of measurements taken with the UUT, e_{res} is the resolution error of the UUT and e_{op} is operator bias associated with the use of the UUT

$$\begin{aligned} e_{rep} &= e_{UUT,rep} \\ e_{res} &= e_{UUT,res} \\ e_{op} &= e_{UUT,op} \, . \end{aligned} \tag{24}$$

The error source e_{other} may need to include mixed contributions as described in Scenario 1.

Substituting Eq. (23) in Eq. (21) and rearranging gives

$$\delta = e_{UUT,b} + e_{MTE,b} + e_{rep} + e_{res} + e_{op} + e_{other} \tag{25}$$

where e_{rep}, e_{res}, and e_{op} are defined in Eq. (24).

As before, we obtain an expression that is separable into a measurement δ of the UUT bias, $e_{UUT,b}$ and an error ε_{cal} given by

$$\varepsilon_{cal} = e_{MTE,b} + e_{rep} + e_{res} + e_{op} + e_{other} \, . \tag{26}$$

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[6] Cases where the UUT measures the value of a metered or other MTE attribute exhibiting a displayed value are covered later as special instances of Scenario 4.

By Eq. (5), the uncertainty in the $e_{UUT,b}$ estimate in Eq. (25) is

$$u_{cal} = \sqrt{\text{var}(\varepsilon_{cal})}\ , \tag{27}$$

where

$$\text{var}(\varepsilon_{cal}) = u_{MTE,b}^2 + u_{rep}^2 + u_{res}^2 + u_{op}^2 + u_{other}^2\ . \tag{28}$$

Scenario 3: MTE and UUT Output Comparison (Comparator Scenario)

In this scenario, a device called a "comparator" is used to compare UUT and MTE values where both the UUT and the MTE provide an output value or stimulus. It is worthwhile to consider the following procedure:

1. The MTE is placed in the comparator.
2. The comparator indication or reading y is noted. This indication or reading is taken to correspond to the MTE nominal or reading value y_n.
3. The MTE is removed and the UUT is placed in the comparator.
4. The comparator indication or reading x is noted.
5. The difference δ is calculated, where

$$\delta = x - y \tag{29}$$

is taken to be a measurement of the UUT bias $e_{UUT,b}$. The UUT corrected value, denoted x_c, is then given by

$$x_c = y_n + \delta\ . \tag{30}$$

Figure 3. Scenario 3 – Measured values of the UUT and MTE attributes are compared using a comparator. The outputs are the weights of the masses.

In keeping with the basic notation, the indicated value y can be expressed as

$$y = y_{true} + \varepsilon_{MTE,m} \tag{31}$$

and the indicated value x can be written

$$x = x_{true} + \varepsilon_{UUT,m} \tag{32}$$

where $\varepsilon_{MTE,m}$ is the measurement error involved in the use of the comparator to measure the MTE attribute value and $\varepsilon_{UUT,m}$ is the measurement error involved in the use of the comparator to measure the UUT attribute value.

By Eq. (4), we can write

$$y_{true} = y_n + e_{MTE,b} \tag{33}$$

and

$$x_{true} = x_n + e_{UUT,b} \tag{34}$$

Substituting Eq. (33) in Eq. (31) gives

$$y = y_n + e_{MTE,b} + \varepsilon_{MTE,m} \tag{35}$$

and substituting Eq. (34) in Eq. (32) yields

$$x = x_n + e_{UUT,b} + \varepsilon_{UUT,m} \, . \tag{36}$$

Using Eqs. (35) and (36) in Eq. (29), we can write

$$\begin{aligned} \delta &= x - y \\ &= x_n - y_n + e_{UUT,b} - e_{MTE,b} + (\varepsilon_{UUT,m} - \varepsilon_{MTE,m}), \end{aligned}$$

so that

$$e_{UUT,b} = \delta - (x_n - y_n) + e_{MTE,b} - (\varepsilon_{UUT,m} - \varepsilon_{MTE,m}) \, . \tag{37}$$

In most calibrations involving comparators $x_n = y_n$ and Eq. (37) becomes [7]

$$e_{UUT,b} = \delta + e_{MTE,b} - (\varepsilon_{UUT,m} - \varepsilon_{MTE,m}) \, . \tag{38}$$

Then, as with other scenarios, we have by Eq. (38), a measured deviation δ and a calibration process error ε_{cal}:

$$\begin{aligned} \delta &= e_{UUT,b} - e_{MTE,b} + (\varepsilon_{UUT,m} - \varepsilon_{MTE,m}) \\ &= e_{UUT,b} + \varepsilon_{cal} \, , \end{aligned} \tag{39}$$

where

$$\varepsilon_{cal} = (\varepsilon_{UUT,m} - \varepsilon_{MTE,m}) - e_{MTE,b} \, . \tag{40}$$

Letting $e_{c,b}$ represent the bias of the comparator, $\varepsilon_{MTE,m}$ is given by[8]

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[7] To accommodate cases where $y_n \neq x_n$, δ is redefined as

$$\delta = (x - x_n) - (y - y_n) \, .$$

As an example where $x_n \neq y_n$, consider a case where the MTE is a 2 cm gage block and the UUT is a 1 cm gage block. Suppose that the comparator readings for the MTE and UUT are 2.10 cm and 0.99 cm, respectively. Then

$$\delta = (0.99 - 1.0) - (2.10 - 2.0) = -0.110 \text{ cm} \, ,$$

and, using Eq. (30), we have

$$x_c = 2.0 \text{ cm} + (0.99 - 2.10) \text{ cm} = (2.0 - 1.11) \text{ cm} = 0.89 \text{ cm}.$$

$$\varepsilon_{MTE,m} = e_{c,b} + e_{MTE,rep} + e_{MTE,res} + e_{MTE,op} + e_{MTE,other} . \tag{41}$$

and $\varepsilon_{UUT,m}$ is

$$\varepsilon_{UUT,m} = e_{c,b} + e_{UUT,rep} + e_{UUT,res} + e_{UUT,op} + e_{UUT,other} . \tag{42}$$

By Eqs. (39) and (5), the measurement uncertainty in δ is obtained from

$$u_{cal} = \sqrt{\text{var}(\varepsilon_{cal})} ,$$

where

$$\text{var}(\varepsilon_{cal}) = u_{MTE,b}^2 + u_{rep}^2 + u_{res}^2 + u_{op}^2 + u_{other}^2 . \tag{43}$$

In this scenario,

$$\begin{aligned} u_{MTE,b}^2 &= \text{var}(-e_{MTE,b}) \\ u_{rep}^2 &= \text{var}(e_{UUT,rep} - e_{MTE,rep}) = u_{MTE,rep}^2 + u_{UUT,rep}^2 \\ u_{res}^2 &= \text{var}(e_{UUT,res} - e_{MTE,res}) = u_{MTE,res}^2 + u_{UUT,res}^2 \\ u_{op}^2 &= \text{var}(e_{UUT,op} - e_{MTE,op}) = u_{MTE,op}^2 + u_{UUT,op}^2 - 2\rho_{op} u_{MTE,op} u_{UUT,op} \end{aligned} \tag{44}$$

and

$$u_{other}^2 = \text{var}(e_{UUT,other} - e_{MTE,other}) = u_{MTE,other}^2 + u_{UUT,other}^2 - 2\rho_{other} u_{MTE,other} u_{UUT,other} , \tag{45}$$

where ρ_{other} represents the correlation, if any, between $e_{MTE,other}$ and $e_{UUT,other}$.

Scenario 4: The MTE and UUT Measure a Common Artifact

In this scenario, both the MTE and UUT measure the value of a common artifact, where the artifact provides an output or stimulus. The measurements are made and recorded separately. An example of this scenario is the calibration of a thermometer (UUT) using a temperature reference (MTE), where both the thermometer and the temperature reference are placed in an oven and the temperatures measured by each are recorded.

We let T denote the true value of the artifact and write the measurement equation as

$$x = T + \varepsilon_{UUT,m} , \tag{46}$$

and

$$y = T + \varepsilon_{MTE,m} , \tag{47}$$

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[8] In many comparator calibrations, the comparator device is made up of two measurement arms and a meter or other indicator. The UUT and the MTE are placed in different arms of the comparator and the difference between the values is displayed by the indicating device. In such cases, if the UUT and MTE swap locations, and the average of the differences is recorded, then bias cancellation is achieved as in Eq. (43). Of course, the Eqs. (41) and (42) would need to be modified to accommodate any additional measurement process errors, such as additional contributions due to comparator resolution error.

If this swapping procedure is not followed, the comparator bias is not cancelled and the applicable scenario becomes a variation of Scenario 1 or 2 in which the comparator, taken in aggregate, is treated as the MTE, with the reference item, indicating device, comparator arms, etc. acting as components. The estimation of the bias uncertainty of the aggregate MTE is the subject of multivariate uncertainty analysis, described in Annex 3.

where $\varepsilon_{UUT,m}$ is the measurement process error for the UUT measurement of the artifact's value and $\varepsilon_{MTE,m}$ is the measurement process error for the MTE measurement of same.

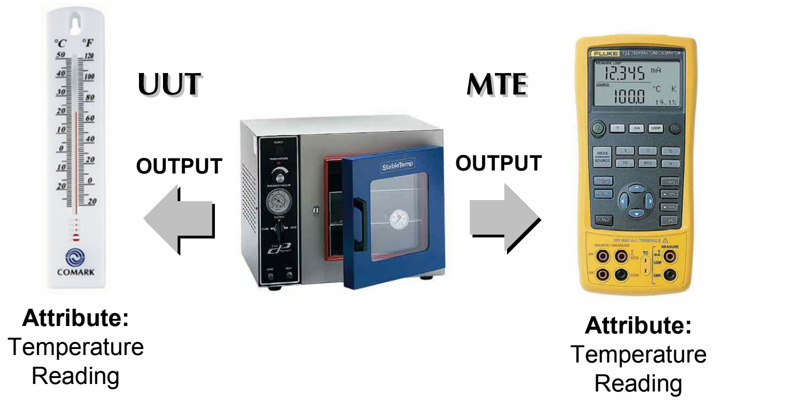

Figure 4. Scenario 4 – The UUT and the MTE measure a common artifact. The output is the temperature of an oven.

These errors are given by

$$\varepsilon_{UUT,m} = e_{UUT,b} + e_{UUT,rep} + e_{UUT,res} + e_{UUT,op} + e_{UUT,other} \tag{48}$$

and

$$\varepsilon_{MTE,m} = e_{MTE,b} + e_{MTE,rep} + e_{MTE,res} + e_{MTE,op} + e_{MTE,other} \,. \tag{49}$$

Substituting these expressions in Eqs. (46) and (47) gives

$$x = T + e_{UUT,b} + e_{UUT,rep} + e_{UUT,res} + e_{UUT,op} + e_{UUT,other} \tag{50}$$

and

$$y = T + e_{MTE,b} + e_{MTE,rep} + e_{MTE,res} + e_{MTE,op} + e_{MTE,other} \,. \tag{51}$$

Defining

$$\delta = x - y \,, \tag{52}$$

these expressions yield

$$\delta = e_{UUT,b} + \varepsilon_{cal} \,. \tag{53}$$

where

$$\begin{aligned}\varepsilon_{cal} = {} & (e_{UUT,rep} - e_{MTE,rep}) + (e_{UUT,res} - e_{MTE,res}) \\ & + (e_{UUT,op} - e_{MTE,op}) + (e_{UUT,other} - e_{MTE,other}) - e_{MTE,b} \,.\end{aligned} \tag{54}$$

By Eq. (5), the measurement uncertainty is again given by

$$u_{cal} = \sqrt{\text{var}(\varepsilon_{cal})} \,, \tag{55}$$

where

$$\text{var}(\varepsilon_{cal}) = u_{MTE,b}^2 + u_{rep}^2 + u_{res}^2 + u_{op}^2 + u_{other}^2 \,, \tag{56}$$

and

$$
\begin{aligned}
u_{MTE,b}^2 &= \mathrm{var}(-e_{MTE,b}) \\
u_{rep}^2 &= \mathrm{var}(e_{UUT,rep} - e_{MTE,rep}) = u_{UUT,rep}^2 + u_{MTE,rep}^2 \\
u_{res}^2 &= \mathrm{var}(e_{UUT,res} - e_{MTE,res}) = u_{UUT,res}^2 + u_{MTE,res}^2 \\
u_{op}^2 &= \mathrm{var}(e_{UUT,op} - e_{MTE,op}) = u_{UUT,op}^2 + u_{MTE,op}^2 - 2\rho_{op} u_{UUT,op} u_{MTE,op} ,
\end{aligned}
\qquad (57)
$$

and

$$
u_{other}^2 = \mathrm{var}(e_{UUT,other} - e_{MTE,other}) = u_{UUT,other}^2 + u_{MTE,other}^2 - 2\rho_{other} u_{UUT,other} u_{MTE,other} , \qquad (58)
$$

where, again, ρ_{other} represents a correlation between $e_{MTE,other}$ and $e_{UUT,other}$.

Scenario 4 Special Cases

There are two special cases of Scenario 4 that may be thought of as variations of Scenarios 1 and 2. Both cases are accommodated by the Scenario 4 definitions and expressions developed above.

Case 1: The MTE measures the UUT and both the MTE and UUT provide a metered or other displayed output.

In this case, the common artifact is the UUT attribute, consisting of a "stimulus" embedded in the UUT. An example would be a UUT voltage source whose output is indicated by a digital display and is measured using an MTE voltmeter.

Case 2: The UUT measures the MTE and both the MTE and UUT provide a metered or other displayed output.

In this case, the common artifact is the MTE attribute, consisting of a "stimulus" embedded in the MTE. An example would be an MTE voltage source whose output is indicated by a digital display and is measured using a UUT voltmeter.

Uncertainty Analysis Examples

Four scenarios have been discussed that yield expressions for calibration uncertainty that are useful for risk analysis. In all scenarios and cases, the calibration result is expressed as

$$
\delta = e_{UUT,b} + \varepsilon_{cal} ,
$$

and the calibration uncertainty is given by

$$
u_{cal} = \sqrt{\mathrm{var}(\varepsilon_{cal})} .
$$

Examples illustrating the application of the four calibration scenarios to uncertainty analysis are provided below.

Scenario 1: The MTE Measures the UUT Attribute Value

In this scenario, the measurement result is $\delta = y - x_n$ and ε_{cal} is expressed in Eq. (9). The example for this scenario consists of calibrating a 30 gm mass with a precision balance. Local gravity is considered to be constant during the measurement process. Multiple measurements of the UUT mass are taken and the sample statistics are computed to be

Sample Mean = 30.000047 gm

Standard Deviation	=	1.15×10^{-5} gm
Uncertainty in the Mean	=	6.64×10^{-6} gm
Sample Size	=	3

The measurement result is $\bar{\delta}$ = (30.000047 – 30) gm = 4.7×10^{-5} gm. However, the measurements are not taken in a vacuum, so the buoyancy of displaced air can introduce measurement error. The balance is calibrated with calibration weights with density of 8.0 gm/cm^3. The air buoyancy correction is

$$\bar{y}_{corr} = \bar{y} \times \frac{(1-\rho_{air}/8.0)}{(1-\rho_{air}/\rho_{UUT})}$$

where $\bar{y}$ is the sample mean, ρ_{air} is the local air density and ρ_{UUT} is the density of the UUT mass. For this analysis, we will assume that $\rho_{air} = 1.2 \times 10^{-3}$ gm/cm^3 and $\rho_{UUT} = 8.4$ gm/cm^3. The corrected sample mean is computed to be

$$\begin{aligned}\bar{y}_{corr} &= 30.000047\,\text{gm} \times \frac{(1-0.0012/8.0)}{(1-0.0012/8.4)} = 30.000047\,\text{gm} \times \frac{(1-0.00015)}{(1-0.00014)} \\ &= 30.000047\,\text{gm} \times \frac{0.99985}{0.99986} = 30.000047\,\text{gm} \times 0.99999 \\ &= 29.99975\,\text{gm}\end{aligned}$$

and the corrected calibration result is $\bar{\delta}_{corr}$ = (29.99975 – 30) gm = – 2.5×10^{-4} gm.

In the mass calibration scenario, we must account for the following measurement process errors:

- Bias in the precision balance, $e_{MTE,b}$.
- Repeatability, $e_{MTE,rep}$.
- Error due to the digital resolution of the balance, $e_{MTE,res}$.
- Environmental factors error resulting from the buoyancy correction, e_{env}.

The error in $\bar{\delta}_{corr}$ is

$$\varepsilon_{cal} = e_{MTE,b} + e_{MTE,rep} + e_{MTE,res} + e_{env}$$

where

$$\begin{aligned}e_{env} &= e_{UUT,env} \\ &= c_1 e_{\rho_{air}} + c_2 e_{\rho_{UUT}}\end{aligned}$$

and $e_{\rho_{air}}$ and $e_{\rho_{UUT}}$ are the errors in the air and UUT densities, respectively. The coefficients c_1 and c_2 are sensitivity coefficients that determine the relative contribution of the errors $e_{\rho_{air}}$ and $e_{\rho_{UUT}}$ to the total error e_{env}. The sensitivity coefficients are defined below.[9]

[9] Guidance on the development of multivariate error models is provided in NCSLI RP-12-2008.

$$c_1 = \frac{\partial \overline{y}_{corr}}{\partial \rho_{air}} = \frac{\overline{y}}{1-\rho_{air}/\rho_{UUT}} \times \left[\frac{1-\rho_{air}/8.0}{\rho_{UUT}\left(1-\rho_{air}/\rho_{UUT}\right)} - \frac{1}{8.0} \right]$$

$$= \frac{30.000047\,\text{gm}}{1-0.0012/8.4} \times \left[\frac{1-0.0012/8.0}{8.4\,\text{gm/cm}^3 \times \left(1-0.0012/8.4\right)} - \frac{1}{8.0\,\text{gm/cm}^3} \right]$$

$$= \frac{30.000047\,\text{gm}}{0.99986} \times \left[0.11905\,\text{cm}^3/\text{gm} - 0.125\,\text{cm}^3/\text{gm} \right]$$

$$= 30.0042\ \text{gm} \times \left(-0.0060\,\text{cm}^3/\text{gm} \right) = -0.1785\,\text{cm}^3$$

$$c_2 = \frac{\partial \overline{y}_{corr}}{\partial \rho_{UUT}} = -\overline{y} \times \frac{\rho_{air}}{\rho_{UUT}^2} \times \frac{1-\rho_{air}/8.0}{\left(1-\rho_{air}/\rho_{UUT}\right)^2}$$

$$= -30.000047\text{gm} \times \frac{0.0012}{\left(8.4\right)^2}\text{cm}^3/\text{gm} \times \frac{1-0.0012/8.0}{\left(1-0.0012/8.4\right)^2}$$

$$= -30.000047\text{gm} \times 0.000017\,\text{cm}^3/\text{gm} \times \frac{0.99985}{0.99971} = -5.1 \times 10^{-4}\,\text{cm}^3$$

The uncertainty in $\overline{\delta}_{corr}$ is

$$u_{cal} = \sqrt{\text{var}(\varepsilon_{cal})}$$

$$= \sqrt{\text{var}(e_{MTE,b}) + \text{var}(e_{MTE,rep}) + \text{var}(e_{MTE,res}) + \text{var}(c_1 e_{\rho_{air}} + c_2 e_{\rho_{UUT}})}$$

$$= \sqrt{u_{MTE,b}^2 + u_{MTE,rep}^2 + u_{MTE,res}^2 + c_1^2 u_{\rho_{air}}^2 + c_2^2 u_{\rho_{UUT}}^2 + 2c_1 c_2 \rho_{env} u_{MTE,env} u_{UUT,env}}\,.$$

The correlation coefficient ρ_{env} accounts for any correlation between $e_{\rho_{air}}$ and $e_{\rho_{UUT}}$. The correlation coefficient can range in value from – 1 to +1. In this analysis, the error in the air density is considered to be uncorrelated to the error in the density of the UUT mass. Therefore, $\rho_{env} = 0$ and the uncertainty u_{cal} can be expressed as

$$u_{cal} = \sqrt{u_{MTE,b}^2 + u_{MTE,rep}^2 + u_{MTE,res}^2 + c_1^2 u_{\rho_{air}}^2 + c_2^2 u_{\rho_{UUT}}^2}$$

$$= \sqrt{u_{MTE,b}^2 + u_{MTE,rep}^2 + u_{MTE,res}^2 + \left(c_1 u_{\rho_{air}}\right)^2 + \left(c_2 u_{\rho_{UUT}}\right)^2}$$

The distributions, limits, confidence levels and standard uncertainties for each error source are summarized in Table 2.

Table 2 Summary of Scenario 1 Uncertainty Estimates

Error	Error Limits	Confidence Level (%)	Error Distribution	Deg. of Freedom	Analysis Type	Standard Uncertainty
$e_{MTE,b}$	± 0.12 gm	95.00	Normal	Infinite	B	6.12×10^{-2} gm
e_{rep}			Student's t	2	A	6.64×10^{-6} gm
e_{res}	± 0.005 gm	100.00	Uniform	Infinite	B	2.9×10^{-3} gm
$e_{\rho_{air}}$	$\pm 3.6 \times 10^{-5}$ gm/cm^3	95.00	Normal	Infinite	B	1.84×10^{-6} gm/cm^3
$e_{\rho_{UUT}}$	± 0.15 gm/cm^3	95.00	Normal	Infinite	B	0.077 gm/cm^3

Using the data in Table 2, the uncertainty in $\bar{\delta}_{corr}$ is computed to be

$$u_{cal} = \sqrt{\begin{array}{l}\left(6.12\times10^{-2}\right)^2 + \left(6.64\times10^{-6}\right)^2 + \left(2.9\times10^{-3}\right)^2 + \left(-0.1785\times1.84\times10^{-6}\right)^2 \\ +\left(-5.1\times10^{-4}\times0.077\right)^2\end{array}} \text{ gm}$$

$$= \sqrt{3.75\times10^{-3} + 4.41\times10^{-11} + 8.41\times10^{-6} + 1.81\times10^{-10} + 1.08\times10^{-13}} \text{ gm}$$

$$\cong \sqrt{3.75\times10^{-3}} \text{ gm}$$

$$\cong 6.13\times10^{-2} \text{ gm.}$$

Using the Welch-Satterthwaite relation, the combined degrees of freedom is computed to be infinite.

Scenario 2: The UUT Measures the MTE Attribute Value

In this scenario, the measurement result is $\delta = x - y_n$ and ε_{cal} is given in Eq. (26). The example for this scenario consists of calibrating an analog micrometer with a 10 mm gage block reference. During the micrometer calibration, the lab temperature was determined to be 24 °C ± 1 °C. Multiple readings of the 10 mm gage block length are taken with the micrometer under laboratory environmental conditions of 24 °C ± 1 °C. The sample statistics are computed to be

Sample Mean	=	9.999 mm
Standard Deviation	=	21.7 μm
Uncertainty in Mean	=	7.7 μm
Sample Size	=	10

The measurement result is $\bar{\delta}$ = (9.999 – 10) mm = –1 μm. However, both the micrometer reading and the gage block length must be corrected to the standard reference temperature of 20 °C. In this example, the gage block steel has a coefficient of thermal expansion of 11.5×10^{-6}/°C and the micrometer has a coefficient thermal expansion of 5.6×10^{-6}/deg °C. For the purposes of this analysis, we will assume that these thermal expansion coefficients are constants.

The net effect of thermal expansion on the measurement result $\bar{\delta}$ is

$$\bar{\delta}_{env} = \bar{\delta}_{UUT,env} - \bar{\delta}_{MTE,env}$$

where $\overline{\delta}_{UUT,env}$ and $\overline{\delta}_{MTE,env}$ represent thermal expansion of the micrometer and gage block dimensions, respectively. The net length expansion is computed from the temperature difference ΔT, the average measured length $\overline{x}$, the coefficient of thermal expansion for the gage block α_{MTE} and the coefficient of thermal expansion for the micrometer α_{UUT}.

$$\begin{aligned}\overline{\delta}_{env} &= \Delta T \times \overline{x} \times (\alpha_{UUT} - \alpha_{MTE}) \\ &= 4\,°\text{C} \times 9.999\,\text{mm} \times (5.6 - 11.5) \times 10^{-6}\,/°\text{C} \\ &= -2.36 \times 10^{-4}\,\text{mm} = -0.236\,\mu\text{m}\end{aligned}$$

The corrected calibration result $\overline{\delta}_{corr}$ is computed to be

$$\begin{aligned}\overline{\delta}_{corr} &= \overline{\delta} + \overline{\delta}_{env} \\ &= -(1 + 0.236)\,\mu\text{m} \\ &= -1.24\,\mu\text{m}\end{aligned}$$

In the micrometer calibration scenario, we must account for the following measurement process errors:

- Bias in the value of the 10 mm gage block length, $e_{MTE,b}$.
- Error associated with the repeat measurements taken, $e_{UUT,rep}$.
- Error associated with the analog resolution of the micrometer, $e_{UUT,res}$.
- Operator bias resulting from his/her perception of the analog readings, $e_{UUT,op}$.
- Environmental factors error resulting from the thermal expansion correction, e_{env}.

The error in $\overline{\delta}_{corr}$ is

$$\varepsilon_{cal} = e_{MTE,b} + e_{UUT,rep} + e_{UUT,res} + e_{UUT,op} + e_{env}$$

where

$$e_{env} = e_{UUT,env} - e_{MTE,env}$$

and $e_{UUT,env}$ and $e_{MTE,env}$ are the errors in the micrometer and gage block length corrections, respectively.

The uncertainty in $\overline{\delta}_{corr}$ is

$$
\begin{aligned}
u_{cal} &= \sqrt{\text{var}(\varepsilon_{cal})} \\
&= \sqrt{\text{var}\left(e_{MTE,b}\right) + \text{var}\left(e_{UUT,rep}\right) + \text{var}\left(e_{UUT,res}\right) + \text{var}\left(e_{UUT,op}\right) + \text{var}\left(e_{UUT,env} - e_{MTE,env}\right)} \\
&= \sqrt{\begin{aligned} &\text{var}\left(e_{MTE,b}\right) + \text{var}\left(e_{UUT,rep}\right) + \text{var}\left(e_{UUT,res}\right) + \text{var}\left(e_{UUT,op}\right) + \text{var}\left(e_{UUT,env}\right) + \text{var}\left(e_{MTE,env}\right) \\ &- 2\rho_{env}\sqrt{\text{var}\left(e_{UUT,env}\right)\text{var}\left(e_{MTE,env}\right)} \end{aligned}} \\
&= \sqrt{u_{MTE,b}^2 + u_{UUT,rep}^2 + u_{UUT,res}^2 + u_{UUT,op}^2 + u_{UUT,env}^2 + u_{MTE,env}^2 - 2\rho_{env}u_{UUT,env}u_{MTE,env}}\ .
\end{aligned}
$$

The correlation coefficient ρ_{env} accounts for any correlation between the environmental correction errors. In this analysis, both the micrometer and gage block length expansion corrections will err in the same direction and by a constant proportional amount. Therefore, a correlation coefficient of +1 should apply and the uncertainty u_{cal} can be expressed as

$$
\begin{aligned}
u_{cal} &= \sqrt{u_{MTE,b}^2 + u_{UUT,rep}^2 + u_{UUT,res}^2 + u_{UUT,op}^2 + u_{UUT,env}^2 + u_{MTE,env}^2 - 2u_{UUT,env}u_{MTE,env}} \\
&= \sqrt{u_{MTE,b}^2 + u_{UUT,rep}^2 + u_{UUT,res}^2 + u_{UUT,op}^2 + \left(u_{UUT,env} - u_{MTE,env}\right)^2}\ .
\end{aligned}
$$

The distributions, limits, confidence levels and standard uncertainties for each error source are summarized in Table 3.

Table 3 Summary of Scenario 2 Uncertainty Estimates

Error	Error Limits (μm)	Confidence Level (%)	Error Distribution	Degrees of Freedom	Analysis Type	Standard Uncertainty (μm)
e_{rep}			Student's t	9	A	7.7
e_{res}	± 5.0	95.00	Normal	Infinite	B	2.6
e_{op}	± 5.0	95.00	Normal	Infinite	B	2.6
$e_{UUT,env}$	± 0.056	95.00	Normal	Infinite	B	0.029
$e_{MTE,env}$	± 0.115	95.00	Normal	Infinite	B	0.059
$e_{MTE,b}$	+ 0.18, -0.13	90.00	Lognormal	Infinite	B	0.09

Using the data in Table 3, the uncertainty in $\bar{\delta}_{corr}$ is

$$
\begin{aligned}
u_{cal} &= \sqrt{(7.7)^2 + (2.6)^2 + (2.6)^2 + (0.029 - 0.056)^2 + (0.09)^2}\ \mu\text{m} \\
&= \sqrt{59.29 + 6.76 + 6.76 + 0.007 + 0.0081}\ \mu\text{m} \\
&= \sqrt{72.82}\ \mu\text{m} \\
&= 8.53\ \mu\text{m}
\end{aligned}
$$

Using the Welch-Satterthwaite relation, the combined degrees of freedom is computed to be 14.

Scenario 3: MTE and UUT Attribute Values are Compared

In this scenario, the measurement result is $\delta = x - y$ and ε_{cal} is expressed in Eq. (40). The example for this scenario consists of calibrating an end gauge, with a nominal length of 50 mm, using an end gauge standard of the same nominal length. The calibration process consists of measuring and recording the difference between the two end gauges using a comparator apparatus.

In this case, we are measuring the difference in the lengths of the two end gauges. The sample statistics are computed to be

Sample Mean	=	215 nm
Standard Deviation	=	9.7 nm
Uncertainty in Mean	=	4.33 nm
Sample Size	=	5

and the measurement result is $\bar{\delta} = 215\,\text{nm}$. The temperature for both gage blocks during calibration is 19.9 °C ± 0.5 °C. Consequently, the calibration result must be corrected to the standard reference temperature of 20 °C. The corrected calibration result $\bar{\delta}_{corr}$ is computed from

$$\begin{aligned}\bar{\delta}_{corr} &= \bar{\delta} + \bar{\delta}_{env} \\ &= \bar{\delta} + \bar{\delta}_{UUT,env} - \bar{\delta}_{MTE,env}\end{aligned}$$

where

$\bar{\delta}_{UUT,env} = \Delta T \times \bar{x} \times \alpha_{UUT}$	=	thermal expansion of the UUT end gage
$\bar{\delta}_{MTE,env} = \Delta T \times \bar{y} \times \alpha_{MTE}$	=	thermal expansion of the MTE end gage
$\bar{\delta}$	=	$\overline{x-y}$, the average difference between UUT and MTE end gage lengths during calibration
α_{UUT}	=	coefficient of thermal expansion for the UUT end gauge
α_{MTE}	=	coefficient of thermal expansion for the MTE end gauge
ΔT	=	difference in the temperature of the end gauge from the 20 °C

For the purposes of this example, we will assume that $\alpha_{UUT} = \alpha_{MTE} = \alpha = 11.5 \times 10^{-6}$/°C. Therefore, $\bar{\delta}_{corr}$ can be expressed as

$$\begin{aligned}\bar{\delta}_{corr} &= \bar{\delta} + \alpha\Delta T\left(\bar{x} - \bar{y}\right) \\ &= \bar{\delta} + \alpha\Delta T\bar{\delta} \\ &= \bar{\delta}\left(1 + \alpha\Delta T\right)\end{aligned}$$

and is computed to be

$$\begin{aligned}\bar{\delta}_{corr} &= 215\,\text{nm}\left(1+0.1\,°\text{C}\times 11.5\times 10^{-6}\,/°\text{C}\right) \\ &= 215\,\text{nm}\left(1+1.15\times 10^{-6}\right) \\ &\cong 215\,\text{nm},\end{aligned}$$

and the corrected value for this example is

$$\begin{aligned} x_c &= y_n + \bar{\delta}_{corr} \\ &= y_n + \bar{\delta}(1+\alpha T) \\ &= 50 \text{ mm} + 215 \text{ μm} \\ &\cong 50.0002 \text{ mm.} \end{aligned}$$

The error in the estimate of x_c includes the following measurement process errors:

- Bias in the value of the 50 mm end gage standard length, $e_{MTE,b}$.
- Bias of the comparator, $e_{c,b}$
- Error associated with the repeat measurements taken, e_{rep}.
- Digital Resolution error for the comparator, e_{res}.
- Environmental factors error resulting from the thermal expansion correction, e_{env}.

The combined calibration error in $\bar{\delta}_{corr}$ is, by Eq. (40),

$$\varepsilon_{cal} = (\varepsilon_{UUT,m} - \varepsilon_{MTE,m}) - e_{MTE,b}$$

where, by Eqs. (41) and (42),

$$\varepsilon_{MTE,m} = e_{c,b} + e_{MTE,rep} + e_{MTE,res} + e_{MTE,env}$$

and

$$\varepsilon_{UUT,m} = e_{c,b} + e_{UUT,rep} + e_{UUT,res} + e_{UUT,env}$$

where $e_{c,b}$ represents the bias of the comparator. Writing the expression for e_{cal} as

$$\varepsilon_{cal} = e_{rep} + e_{res} + e_{env} - e_{MTE,b}$$

we have

$$e_{rep} = e_{UUT,rep} - e_{MTE,rep} = e_{\bar{\delta},rep}$$

$$e_{res} = e_{UUT,res} - e_{MTE,res}$$

$$e_{env} = e_{UUT,env} - e_{MTE,env} = e_{\bar{\delta},env}$$

where

$$e_{\bar{\delta},env} = c_1 e_{\Delta T} + c_2 e_{\alpha}.$$

The sensitivity coefficients c_1 and c_2 are given by

$$\begin{aligned} c_1 &= \frac{\partial \bar{\delta}_{env}}{\partial \Delta T} = \alpha \bar{\delta} \\ &= 11.5 \times 10^{-6} / {}^\circ\text{C} \times 215 \text{nm} \\ &= 2.47 \times 10^{-3} \text{nm} / {}^\circ\text{C} \end{aligned} \quad \text{and} \quad \begin{aligned} c_2 &= \frac{\partial \bar{\delta}_{env}}{\partial \alpha} = \Delta T \bar{\delta} \\ &= 0.1^\circ\text{C} \times 215 \text{nm} \\ &= 21.5^\circ\text{C-nm} \end{aligned}$$

The uncertainty in $\bar{\delta}_{corr}$ is

$$
\begin{aligned}
u_{cal} &= \sqrt{\mathrm{var}(\varepsilon_{cal})} \\
&= \sqrt{\mathrm{var}\left(e_{\bar{\delta},rep}\right) + \mathrm{var}\left(e_{UUT,res} - e_{MTE,res}\right) + \mathrm{var}\left(c_1 e_{\Delta T} + c_2 e_{\alpha}\right) + \mathrm{var}\left(-e_{MTE,b}\right)} \\
&= \sqrt{u_{\bar{\delta},rep}^2 + u_{UUT,res}^2 + u_{MTE,res}^2 - 2\rho_{res} u_{UUT,res} u_{MTE,res} + c_1^2 u_{\Delta T}^2 + c_2^2 u_{\alpha}^2 + u_{MTE,b}^2}.
\end{aligned}
$$

The resolution uncertainty for the UUT and MTE are equal to the resolution uncertainty of the comparator, $u_{UUT,res} = u_{MTE,res} = u_{c,res}$. In addition, the resolution error for the UUT and MTE are uncorrelated, so that $\rho_{res} = 0$. Therefore, the uncertainty u_{cal} can be expressed as

$$
u_{cal} = \sqrt{u_{\bar{\delta},rep}^2 + 2u_{c,res}^2 + c_1^2 u_{\Delta T}^2 + c_2^2 u_{\alpha}^2 + u_{MTE,b}^2}\ .
$$

The distributions, limits, confidence levels and standard uncertainties for each error source are summarized in Table 4.

Table 4 Summary of Scenario 3 Uncertainty Estimates

Error	Error Limits	Confidence Level (%)	Error Distribution	Degrees of Freedom	Analysis Type	Standard Uncertainty (nm)
$e_{\bar{\delta},rep}$			Student's t	5	A	4.33 μm
$e_{c,res}$	± 1 nm	100.0	Uniform	Infinite	B	0.577 μm
$e_{\Delta T}$	± 0.5 °C	95.00	Normal	Infinite	B	0.255 °C
e_{α}	± 0.5 × 10^{-6} /°C	95.00	Normal	Infinite	B	± 0.255 × 10^{-6} /°C
$e_{MTE,b}$			Student's t	18	B	25 μm

Using the data in Table 4, the uncertainty in $\bar{\delta}_{corr}$ is

$$
\begin{aligned}
u_{cal} &= \sqrt{(4.33)^2 + 2\times(0.577)^2 + (2.47\times10^{-3}\times0.255)^2 + (21.5\times0.255\times10^{-6})^2 + (25)^2}\ \text{nm} \\
&= \sqrt{18.75 + 0.67 + 3.97\times10^{-7} + 3.01\times10^{-5} + 625}\ \mu\text{m} \\
&= \sqrt{644.4}\ \mu\text{m} \\
&= 25.4\ \mu\text{m}.
\end{aligned}
$$

Using the Welch-Satterthwaite relation, the combined degrees of freedom is computed to be 19.

Scenario 4: The MTE and UUT Measure a Common Artifact

For this scenario, both the MTE and UUT measure the value or output of a common artifact. The measurement result is $\delta = x - y$ and ε_{cal} is given in Eq. (54).

The example for this scenario consists of calibrating a digital thermometer at 100 °C using an oven and analog temperature reference. The oven temperature is adjusted using its internal temperature probe and the readings from the thermometer and temperature reference are recorded. This process is repeated several times and the resulting sample statistics are computed to be

Sample Mean, UUT	=	100.01 °C	Sample Mean, MTE	=	100.000 °C
Standard Deviation, UUT	=	0.03 °C	Standard Deviation, MTE	=	0.006 °C
Uncertainty in Mean, UUT	=	0.01 °C	Uncertainty in Mean, MTE	=	0.002 °C
Sample Size	=	9	Sample Size	=	9

and the measurement result is $\bar{\delta} = 100.01 - 100.000 = 0.01°\text{C}$. In the thermometer calibration scenario, we must account for the following measurement process errors:

- Bias of the temperature reference, $e_{MTE,b}$.
- Error due to repeat measurements taken with the temperature reference, $e_{MTE,rep}$.
- Error due to repeat measurements taken with the thermometer, $e_{UUT,rep}$.
- Analog resolution error for the temperature reference, $e_{MTE,res}$.
- Digital resolution error for the thermometer, $e_{UUT,res}$.

The short-term effects of oven stability and uniformity are accounted for in the sample of MTE measurements. If repeat measurements were not collected, then errors due to oven stability and uniformity would be included in the analysis. These would be considered environmental factors errors.

The error in $\bar{\delta}$ is, from Eq. (54),

$$\varepsilon_{cal} = (e_{UUT,rep} - e_{MTE,rep}) + (e_{UUT,res} - e_{MTE,res}) - e_{MTE,b}$$

and the uncertainty in $\bar{\delta}$ is

$$\begin{aligned} u_{cal} &= \sqrt{\text{var}(\varepsilon_{cal})} \\ &= \sqrt{\text{var}(-e_{MTE,b}) + \text{var}(e_{UUT,rep} - e_{MTE,rep}) + \text{var}(e_{UUT,res} - e_{MTE,res})} \\ &= \sqrt{u_{MTE,b}^2 + u_{UUT,rep}^2 + u_{MTE,rep}^2 + u_{UUT,res}^2 + u_{MTE,res}^2}. \end{aligned}$$

The distributions, limits, confidence levels and standard uncertainties for each error source are summarized in Table 5.

Table 5 Summary of Scenario 4 Uncertainty Estimates

Error	Error Limits (°C)	Confidence Level (%)	Error Distribution	Degrees of Freedom	Analysis Type	Standard Uncertainty (°C)
$e_{MTE,b}$			Student's t	29	A,B	0.02
$e_{UUT,rep}$			Student's t	8	A	0.01
$e_{MTE,rep}$			Student's t	8	A	0.002
$e_{UUT,res}$	± 0.005	100.00	Uniform	infinite	B	0.0029
$e_{MTE,res}$	± 0.0025	95.00	Normal	infinite	B	0.0013

Using the data in Table 5, the uncertainty in $\bar{\delta}$ is

$$
\begin{aligned}
u_{cal} &= \sqrt{(0.02)^2 + (0.01)^2 + (0.002)^2 + (0.0029)^2 + (0.0013)^2}\ °\text{C} \\
&= \sqrt{0.0004 + 0.0001 + 0.000004 + 0.000008 + 0.000002}\ °\text{C} \\
&= \sqrt{0.000514}\ °\text{C} \\
&= 0.023\ °\text{C}
\end{aligned}
$$

Using the Welch-Satterthwaite relation, the combined degrees of freedom is computed to be 34.

Measurement Decision Risk Analysis

In each of the scenarios described in this paper, a UUT bias $e_{UUT,b}$, a measurement of this bias δ, and a measurement uncertainty u_{cal} have been described. In calibrating a UUT to determine if it is in- or out-of-tolerance, we face two principal varieties of measurement decision risk; namely, False Accept Risk and False Reject Risk. The former can be expressed in two ways. First, there is the probability that a UUT attribute is both out-of-tolerance and observed to be in-tolerance. Second, there is the probability that a UUT attribute, accepted as being in-tolerance, will be out-of-tolerance. The first alternative is called "unconditional false accept risk" or *UFAR*. The second is called "conditional false accept risk" or *CFAR*.[10]

False reject risk (*FRR*) is the probability that a UUT attribute will be both in-tolerance and perceived as being out-of-tolerance.[11]

UFAR, *CFAR* and *FRR* are computed for each of the scenarios presented in this paper in a companion article titled "Decision Risk Analysis for Alternative Calibration Scenarios" [9].

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[10] *UFAR* and *CFAR* are also referred to respectively as the "probability of a false accept" or *PFA* and the "conditional probability of a false accept" or *CPFA*. In much of the literature, *UFAR* is also referred to as "Consumer's Risk."

[11] False reject risk is sometimes called the "probability of a false reject" or *PFR*.

Nomenclature

Nomenclature used in this paper for the principal quantities is summarized in Table 6. The notation for other quantities can be determined by applying the notation of Table 1.

Table 6. Nomenclature

Quantity	Description
UUT	Unit Under Test. The artifact undergoing calibration.
Attribute	A measurable property of a device, substance or other quantity.
MTE	Measuring or Test Equipment. The measurement reference.
ε_m	The total error in the measurement of the value of an attribute.
$e_{UUT,b}$	(1) The bias of a UUT attribute as received for calibration. (2) The quantity estimated by UUT calibration.
$u_{UUT,b}$	The uncertainty in the bias of a UUT attribute as received for calibration. Equal to the standard deviation of the $e_{UUT,b}$ distribution.
$e_{MTE,b}$	The bias of the MTE attribute used to calibrate the UUT attribute.
$\varepsilon_{UUT,m}$	The error in measurements made with the UUT attribute or the error in measuring the UUT attribute's value with a comparator.
$\varepsilon_{MTE,m}$	The error in measurements made with the MTE attribute or the error in measuring the MTE attribute's value with a comparator.
δ	The result of a UUT calibration, i.e., an estimate of $e_{UUT,b}$ obtained by calibration.
ε_{cal}	The error in δ.
u_{cal}	The uncertainty in ε_{cal}.
x_n	The nominal value of a UUT attribute.
x_{true}	The true value of a UUT attribute.
y_n	The nominal value of an MTE attribute.
y_{true}	The true value of an MTE attribute.
x_c	The value of the UUT attribute indicated by a measurement taken with a comparator.
$e_{c,b}$	The bias in a comparator indication.
UFAR	Unconditional False Accept Risk. The probability that an out-of-tolerance UUT attribute will be observed to be in-tolerance.
CFAR	Conditional False Accept Risk. The probability that an accepted UUT attribute will be out-of-tolerance.
FRR	False Reject Risk. The probability that an in-tolerance UUT attribute will be observed to be out-of-tolerance.

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