Bayesian Risk Analysis

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Using Bayes theorem, methods were developed in the mid to late 1980s that enabled the analysis of false accept risk for UUT parameters, the estimation of both UUT parameter and measurement reference parameter biases, and the uncertainties in these biases. These methods have been referred to as Bayesian risk analysis methods or, simply, Bayesian analysis methods.

In applying Bayesian analysis methods, we will perform a risk analysis for accepting a specific parameter based on *a priori* knowledge and on the results of measuring the parameter during testing or calibration.

These results comprise what is called "post-test" knowledge which, when combined with *a priori* knowledge, allows us to compute the quantities of interest.

Risk Analysis for a Measured Variable

The procedure for applying Bayesian analysis methods to perform risk analysis for a measured parameter is as follows:

1. Assemble all relevant a priori knowledge, such as the tolerance limits for the UUT parameter, the tolerance limits for the measurement reference parameter, the in-tolerance probabilities for each parameter and the uncertainty of the measuring process.
2. Perform a measurement or set of measurements. This may either consist of measuring the UUT parameter with the measurement reference parameter, measuring the measurement reference parameter with the UUT parameter or using both parameters to measure a common artifact.
3. Estimate the UUT parameter and measurement reference parameter biases using Bayesian analysis methods.
4. Compute uncertainties in the bias estimates.
5. Act on the results. Either report the biases and bias uncertainties, along with in-tolerance probabilities for the parameters, or adjust each parameter to correct the estimated biases.

A priori Knowledge

The *a priori* knowledge for a Bayesian analysis may include several kinds of information. For example, if the UUT parameter is the pressure of an automobile tire, such knowledge may include a rigorous projection of the degradation of the tire's pressure as a function of time since the tire was last inflated or a SWAG estimate based on the appearance of the tire's lateral bulge. However *a priori* knowledge is obtained, is should lead to the following quantities:

- Estimates of the uncertainties in the biases of both the UUT parameter and measurement reference parameter. These estimates may be obtained heuristically from containment limits and containment probabilities or by other means, if applicable.
- An estimate of the uncertainty in the measurement process, accounting for all error sources.

Post-Test Knowledge

The post-test knowledge in a Bayesian analysis consists of the results of measurement. As stated earlier, these results may be in the form of a measurement or a set of measurements. The measurements may be the result of readings provided by the measurement reference parameter from measurements of the UUT parameter, readings provided by the UUT parameter from measurements of the measurement reference parameter, or readings provided by both the UUT parameter and measurement reference parameter, taken on a common artifact.

Bias Estimates

UUT parameter and measurement reference parameter biases are estimated using the method described in [1 - 4]. The method encompasses cases where a measurement sample is taken by either the UUT parameter, the measurement reference parameter or both.[1] The variables are

$\bar{y}_s$ - The perceived value of the UUT parameter. This may be (1) the parameter's nominal value, (2) the mean value of a sample of measurements taken on an artifact by the UUT parameter, also measured by the measurement reference parameter, or (3) the mean value of a sample of direct measurements of the measurement reference parameter value.[2]

s_s - UUT parameter sample standard deviation

u_s - UUT parameter bias uncertainty

$\bar{y}_m$ - The perceived value of the measurement reference parameter. This may be (1) the parameter's nominal value, (2) the mean value of a sample of measurements taken on an artifact by the parameter, also measured by the UUT parameter, or (3) the mean value of a sample of direct measurements of the UUT parameter value.

s_m - measurement reference parameter sample standard deviation

u_m - measurement reference parameter bias uncertainty

UUT parameter and measurement reference parameter biases are estimated according to

$$\text{UUT Parameter Bias} = \frac{(\bar{y}_s - \bar{y}_m)(u_s / u_m)^2}{1 + (u_s / u_m)^2}$$

$$\text{Measurement Reference Parameter Bias} = \frac{(\bar{y}_m - \bar{y}_s)(u_m / u_s)^2}{1 + (u_m / u_s)^2},$$

where

$$u_s = \begin{cases} u_{sb}\,, & \text{when estimating UUT parameter bias} \\ \sqrt{u_{sb}^2 + \dfrac{s_s^2}{n_s} + u_{process}^2}\,, & \text{when estimating measurement reference parameter bias} \end{cases}$$

and

$$u_m = \begin{cases} \sqrt{u_{mb}^2 + \dfrac{s_m^2}{n_m} + u_{process}^2}\,, & \text{when estimating UUT parameter bias} \\ u_{mb}\,,\ \ , & \text{when estimating measurement reference parameter bias} \end{cases}.$$

Bias Uncertainty Estimates

The variable $u_{process}$ is a combination of uncertainties due to process error sources, excluding the random components and the bias uncertainties. The quantities u_{sb} and u_{mb} are *a priori* estimates for UUT parameter and measurement reference parameter bias uncertainties. They are computed from tolerance limit and percent in-tolerance information on the populations of items from which the UUT and measuring unit are drawn.

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[1] Actually, the methodology described in NASA 1342 can also be applied to measurements of a quantity made by any number of independent devices.

[2] The nominal value of a 10 vDC voltage source is an example of case (1). The measurement of ambient air pressure by a pressure gage (UUT parameter) and a dead weight tester (measurement reference parameter) is an example of case (2). The direct measurement of a reference mass (measurement reference parameter) by a precision balance (UUT parameter) is an example of case (3).

For instance, if the percent in-tolerance for the UUT parameter population is 100 × p %, the tolerance limits are L_1 and L_2, and the population's probability density function is $f(x,u_{sb})$, then u_{sb} is determined by inverting the equation

$$p = \int_{L_1}^{L_2} f(x, u_{sb})dx .$$

For example, if x is $N(\mu, u_{sb}^2)$, $L_1 = \mu - L$, and $L_2 = \mu + L$, then

$$u_{sb} = \frac{L}{\Phi^{-1}\left(\frac{1+p}{2}\right)},$$

where $\Phi^{-1}(\cdot)$ is the inverse normal distribution function. Uncertainties in the parameter bias estimates are computed from

$$\text{UUT Parameter Bias Uncertainty} = \frac{r_s^2}{1+r_s^2} u_m ,$$

and

$$\text{Measurement Reference Parameter Bias Uncertainty} = \frac{1}{1+r_s^2} u_s ,$$

where

$$r_s = u_s / u_m .$$

UUT Parameter In-Tolerance Probability

Following measurement, the in-tolerance probability of the UUT parameter may be determined from the relation

$$P(-L \le x \le L) = \Phi(a_+) + \Phi(a_-) - 1$$

where

$$a_{\pm} = \frac{\sqrt{1+r_s^2}\left(L_s \pm \frac{r_s^2}{1+r_s^2} X\right)}{u_s},$$

In this expression, ± L_S are the UUT parameter tolerance limits, and the function $\Phi(a_\pm)$ is defined according to

$$\begin{aligned} \Phi(a_\pm) &= P(x < a_\pm) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a_\pm} e^{-x^2/2} dx . \end{aligned}$$

Appendix 1 - Methodology Development

Imagine that we are calibrating a UUT parameter with symmetric two-sided tolerance limits of ±*L* and an "as received" in-tolerance probability *p*. Assuming a normal distribution for the bias (deviation from nominal) in the parameter, we estimate the *a priori* standard uncertainty for the bias distribution as[3]

$$u_s = \frac{L}{\Phi^{-1}\left(\frac{1+p}{2}\right)},$$

where the function $\Phi^{-1}(\cdot)$ is the inverse of the normal distribution function. Both the normal distribution function and its inverse can be found in statistical textbooks and can be computed with most spreadsheet programs.

We next estimate the total test or calibration process standard uncertainty (including the bias uncertainty of the measurement reference parameter), which we label u_m.

We now label the nominal value of the UUT parameter Y_0, and obtain a measured value of Y.[4] The observed deviation from nominal is therefore given by

$$X = Y_0 - Y.$$

The Parameter Bias Probability Density Function

We now label the actual deviation from nominal for the UUT parameter as ε_0. We can readily obtain the conditional pdf for obtaining an observed (measured) deviation X, given ε_0. Assuming that the conditional probability density function (pdf) for measured values of X is $N(\varepsilon_0, u_m)$, we have

$$f\left(X \mid \varepsilon_0\right) = \frac{1}{\sqrt{2\pi}u_m} e^{-\left(X-\varepsilon_0\right)^2 / 2u_m^2} .$$

where

$$u_m = \sqrt{u_{mb}^2 + \frac{s_m^2}{n_m} + u_{process}^2} .$$

For the analysis at hand, what we seek is the conditional pdf for ε_0, given X, that will, when integrated between the UUT parameter tolerance limits, yield the conditional probability P that the UUT parameter is in-tolerance. Invoking Bayes' theorem, we have

$$f\left(\varepsilon_0 \mid X\right) = \frac{f\left(X \mid \varepsilon_0\right) f\left(\varepsilon_0\right)}{f\left(X\right)},$$

where the prior distribution for ε_0 is given by

$$f\left(\varepsilon_0\right) = \frac{1}{\sqrt{2\pi}u_s} e^{-\varepsilon_0^2 / 2u_s^2} .$$

Combining the above gives the joint probability density function

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[3] For a more detailed treatment, see references [1] - [4].

[4] The variable *Y* is used to represent either a single measurement or the mean value of a sample of measurements.

$$
\begin{aligned}
f\left(X \mid \varepsilon_0\right) f\left(\varepsilon_0\right) &= C \exp\left\{-\frac{1}{2}\left[\frac{\varepsilon_0^2}{u_s^2}+\frac{\left(X-\varepsilon_0\right)^2}{u_m^2}\right]\right\} \\
&= C \exp\left\{-\frac{1}{2u_s^2}\left[\varepsilon_0^2+r_s^2\left(X-\varepsilon_0\right)^2\right]\right\} \\
&= Ce^{-G(X)} \exp\left\{-\frac{1}{2u_s^2}\left[\left(1+r_s^2\right)\left(\varepsilon_0-\frac{r_s^2}{1+r_s^2}X\right)^2\right]\right\},
\end{aligned}
$$

where

$$r_s \equiv u_s / u_m \,,$$

C is a normalization constant, and

$$G(X)=\frac{r_s^2}{1+r_s^2}(X^2/2u_s^2)\,.$$

The pdf $f(X)$ is obtained by integrating the joint distribution over all values of ε_0. To simplify the notation, we define

$$\alpha=\sqrt{1+r_s^2}$$

and

$$\beta=\frac{r_s^2}{1+r_s^2}X\ .$$

Using these expressions in the joint pdf and integrating over ε_0 gives

$$
\begin{aligned}
f\left(X\right) &= Ce^{-G(X)}\int_{-\infty}^{\infty} e^{-\alpha^2\left(\varepsilon_0-\beta\right)^2/2u_s^2}\,d\varepsilon_0 \\
&= Ce^{-G(X)}\frac{\sqrt{2\pi}u_s}{\alpha}\,.
\end{aligned}
$$

Combining this expression with the expression for $G(X)$ yields

$$
\begin{aligned}
f\left(X\right) &= C\frac{\sqrt{2\pi}u_s}{\alpha}\exp\left[-\frac{r_s^2}{1+r_s^2}(X^2/2u_s^2)\right] \\
&= C\frac{\sqrt{2\pi}u_s}{\alpha}\exp\left[-X^2/2u_s^2(\alpha/r_s)^2\right].
\end{aligned}
$$

The constant C is obtained from

$$
\begin{aligned}
C\frac{\sqrt{2\pi}u_s}{\alpha}\int_{-\infty}^{\infty}\exp\left[-X^2/2u_s^2(\alpha/r_s)^2\right]dX &= C\frac{\sqrt{2\pi}u_s}{\alpha}\sqrt{2\pi}(\alpha/r_s)u_s \\
&= 2\pi u_s^2/r_s C \\
&= 1,
\end{aligned}
$$

and

$$C=\frac{r_s}{2\pi u_s^2}\,.$$

Then

$$f\left(X\right) = \frac{1}{\sqrt{2\pi}\sigma_X} \mathrm{e}^{-X^2/2\sigma_X^2} ,$$

where

$$\sigma_X = (\alpha / r_s) u_s$$

Dividing this result into the joint pdf and substituting in the expression for $f(\varepsilon_0 \mid X)$ gives

$$f\left(\varepsilon_0 \mid X\right) = \frac{1}{\sqrt{2\pi}\left(u_s / \alpha\right)} e^{-\left(\varepsilon_0 - \beta\right)^2 / 2\left(u_s / \alpha\right)^2} .$$

UUT Parameter Bias Estimate

From the pdf $f(\varepsilon_0|X)$, it is evident that the UUT parameter bias is estimated by the quantity β

$$\begin{aligned} \text{UUT Parameter Bias} &= \beta \\ &= \frac{r_s^2}{1 + r_s^2} X. \end{aligned}$$

The uncertainty in the bias is given by

$$\begin{aligned} \text{bias uncertainty} &= \sqrt{\mathrm{var}(\beta)} \\ &= \frac{r_s^2}{1 + r_s^2} \sqrt{\mathrm{var}(X \mid \varepsilon_0)} \\ &= \frac{r_s^2}{1 + r_s^2} u_m . \end{aligned}$$

UUT Parameter In-tolerance Probability

The in-tolerance probability for the UUT is obtained by integrating the pdf $f(\varepsilon_0 \mid X)$ over $\mathfrak{L}$

$$\begin{aligned} P &= \frac{1}{\sqrt{2\pi}\left(u_s / \alpha\right)} \int_{-L}^{L} e^{-\left(\varepsilon_0 - \beta\right)^2 / 2\left(u_s / \alpha\right)^2} d\varepsilon_0 \\ &= \Phi\left(\frac{L + \beta}{u_s / \alpha}\right) + \Phi\left(\frac{L - \beta}{u_s / \alpha}\right) - 1 \\ &= \Phi\left(a_+\right) + \Phi\left(a_-\right) - 1 , \end{aligned}$$

where

$$\begin{aligned} a_\pm &= \frac{L \pm \beta}{u_s / \alpha} \\ &= \frac{\sqrt{1 + r_s^2}\left(L \pm \frac{r_s^2}{1 + r_s^2} X\right)}{u_s} , \end{aligned}$$

as before.

Appendix 2 - Developing Guardbands Using Bayesian Analysis

We now develop a baseline procedure for developing guardbands using Bayesian estimates. Following this, we will examine an alternative approach to controlling risks that obviates the need for such guardbands.

The Procedure

The procedure for developing guardbands using Bayesian analysis begins with finding a value of $\beta = \beta_R$ that satisfies

$$\Phi\left(\frac{L+\beta_R}{u_s/\alpha}\right)+\Phi\left(\frac{L-\beta_R}{u_s/\alpha}\right)-1=1-R\,,$$

where R is the maximum allowable false accept risk. Once this value has been found, the guardband limits $\pm A_R$ are obtained from

$$A_R=\frac{1+r_s^2}{r_s^2}\beta_R\,.$$

Why Use Guardbands?

Since the whole point of applying guardbands is to control false accept risk to some specified level, why not simply apply the risk criterion directly? In short, since $1 - P_{in}$ is the computed false accept risk, just set a minimum allowable limit for P_{in} and forget about guardbands. They require extra effort to develop and, if we compute P_{in} using Bayesian methods, they don't really add any benefit.

However, if your quality system requires guardbands, they can be developed using the above procedure.

References

[1] Castrup, H., Intercomparison of Standards: General Case, SAI Comsystems Technical Report, U.S. Navy Contract N00123-83-D-0015, Delivery Order 4M03, March 16, 1984.

[2] Jackson, D.H., Instrument Intercomparison: A General Methodology, Analytical Metrology Note AMN 86-1, U.S. Navy Metrology Engineering Center, NWS Seal Beach, January 1, 1986; and "Instrument Intercomparison and Calibration," Proc. 1987 MSC, Irvine, January 29, 30.

[3] Castrup, H., "Analytical Metrology SPC Methods," Proc. 1991 NCSL Workshop and Symposium, August 1991, Albuquerque.

[4] NASA Metrology Working Group, *Metrology – Calibration and Measurement Process Guidelines*, NASA Ref. Pub. 1342, App. D, June 1994.